What is a Plausible Stress Scenario?*

THOMAS BREUER, GERALD KRENN

Abstract

Plausibility is an important quality criterion for stress test scenarios: scenarios which are highly implausible undermine the credibility of stress tests. In this paper we introduce a measure of plausibility which is applicable under a wide range of distributional assumptions for risk factor changes. We give explicit formulas for the plausibility of scenarios under general elliptical distributions and in the special cases of normally and t-distributed risk factor changes.

1 Introduction

Many banks conduct stress tests for their trading portfolio by considering certain predefined standard scenarios. However, such standardized stress tests do not take into account the composition of the bank's portfolio. As a consequence, it is quite likely that there exist scenarios which are much more harmful to a bank's portfolio than the ones used in standardized stress tests. Therefore, instead of only using standard scenarios, one can search for the scenario with maximum loss (MaxLoss): the worst case scenario.

But in general, MaxLoss will not be finite if all scenarios are considered. It will be impossible to find a market state in which the portfolio has its

---

smallest value, since the loss potential of a portfolio is usually unlimited. A simple example is that of a portfolio which consists only of a written call option: its value will fall without limit as long as the value of the underlying instrument rises. For this reason, not all scenarios will be admitted, but the search for MaxLoss will be restricted to some admissibility domain. Studer (1999) uses the term *trust region* for admissibility domain. One of the nice properties of MaxLoss is the fact that it is a coherent risk measure in the sense of Artzner *et al.* (1999).

Besides size of loss, an important quality criterion for stress scenarios is plausibility. Scenarios which are highly implausible undermine the credibility of stress tests. Even if the size of losses in a stress test is alarming, management will be reluctant to take risk-reducing measures if the scenario leading to such a loss is highly implausible. And rightly so. One way to overcome this shortfall of stress tests is to consider only scenarios above a certain plausibility threshold when calculating MaxLoss. The resulting stress scenarios will satisfy both quality criteria: they lead to heavy losses and they are sufficiently plausible.

This procedure requires a notion of plausibility for stress scenarios. In Section 2 we propose a measure of plausibility which is applicable under a wide range of distributional assumptions for risk factor changes. In Section 3 we give explicit formulas for the plausibility of a scenario and for admissibility domains under elliptical distributions. As special cases we deal with the multivariate normal and the multivariate t-distribution.

2   A Measure of Plausibility for Stress Scenarios

Which scenarios should be considered plausible, which implausible? A first idea might be to call a scenario plausible if it is very close to the present state of the market. Such a concept of plausibility is linked to the size of the move from the present state to the future scenario: the larger the move, the less plausible the scenario. This concept is intuitively appealing and widely used.

However, it is not clear how to measure the "size" of a joint move in several risk factors. If scenario A involves a large move in factor 1 and a small move in factor 2, whereas scenario B involves a small move in factor

1 and a large move in factor 2, is scenario A or scenario B more plausible? Certainly it will not do to take the average over the sizes of single factor moves as size of the joint move. This would neglect dependencies between risk factors which are crucial for measuring the plausibility of joint moves. A scenario in which risk factors move against correlations is not plausible, even if every individual risk factor movement is fairly plausible. For this reason, correlations - or more generally dependencies - have to be taken into account when defining plausibility conditions.

Another kind of plausibility statement is that a certain event is a "once in 100 years event". This is supposed to express that this or a more extreme event occurs once in 100 years. Such a plausibility statement boils down to a statement about a quantile: "This event is located at the $1/(250 \cdot 100)$ quantile." (Assuming the year has 250 trading days and that we are speaking about daily moves.) This suggests that the plausibility of a scenario should be linked to the tail of the scenario. By tail we mean the probability of a more extreme scenario - either on its right side (right tail) or left side (left tail). For an upside move, for example, one might say that a scenario is more plausible if its right tail is larger.

But for two reasons such a tail statement might also be inappropriate as a statement of plausibility. First, we need a measure of plausibility not just for moves in one risk factor, but for joint moves of several factors. And there is a second problem with tail-based measures of plausibility. Consider the density function plotted in Figure 1 on the left hand side. At $x = 1\%$ the density function is lower but the right tail is greater than at $x = 7\%$. Should a move of $+1\%$ now have a higher or lower plausibility than a move of $+7\%$? In this case a tail-based measure of plausibility cannot agree with a density-based measure.

This problem is not related to the fact that the mode of the distribution is not $x = 0\%$. If we set the origin of the x'-axes to the mode we get the density plotted on the right hand side of Figure 1. At $x' = -3\%$ the density is higher than at $x' = +5\%$, but the left tail of $x' = -3\%$ is smaller than the right tail of $x' = +5\%$. Should a move of -3% have higher or lower plausibility than a move of $+5\%$?

In order to formulate a definition of plausibility we need a bit of notation. A scenario is characterized by the values all risk factors have in the market state corresponding to this scenario. These values can be gathered into a vector $\boldsymbol{r} = (r_1, \ldots, r_n)$. Denote by $\boldsymbol{r}_{\mathrm{CM}}$ the current state of the market and by $\Delta \boldsymbol{r}$ the relative change from $\boldsymbol{r}_{\mathrm{CM}}$ to the scenario $\boldsymbol{r}$. This means that the
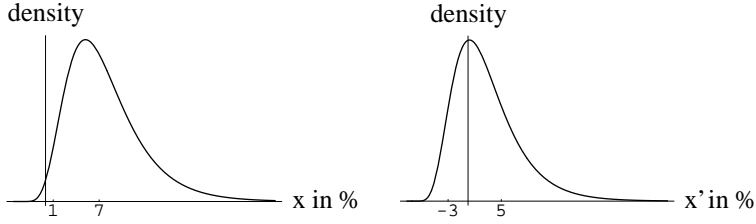
Figure 1: *The density function of relative changes in one risk factor. A tail-based measure of plausibility cannot agree with a density-based measure of plausibility.*
Left: Should a move of $+1\%$ have higher or lower plausibility than a move of $+7\%$?
Right: Should a move of -3% have higher or lower plausibility than a move of $+5\%$?

i-th component of $\Delta \boldsymbol{r}$ is given by $(r_i - r_{\mathrm{CM},i})/r_{\mathrm{CM},i}$.

f shall denote the multivariate density function of these relative risk factor changes over the specified holding period. As we are interested in stress testing, we focus on the unconditional distribution of risk factor changes. The reason for this is that in a stress situation the gain of information about future behavior of risk factors due to conditioning on the recent past is likely to break down. Thus, f is the unconditional density function.

Note the one-to-one correspondence between $\boldsymbol{r}$ and $\Delta \boldsymbol{r}$. This implies in particular that any probability distribution for relative changes of risk factors defines a probability distribution for the risk factors in absolute terms, and vice versa.

Definition 1 *The* plausibility *of a scenario $\boldsymbol{r}_{\mathrm{stress}}$ is the probability of the set*

$$\{\boldsymbol{r} : f(\Delta \boldsymbol{r}) \leq f(\Delta \boldsymbol{r}_{\mathrm{stress}})\}$$

of scenarios to which a move from $\boldsymbol{r}_{\mathrm{CM}}$ has equal or lower density than the move from $\boldsymbol{r}_{\mathrm{CM}}$ to $\boldsymbol{r}_{\mathrm{stress}}$.

By this definition the plausibility of a scenario $\boldsymbol{r}_{\mathrm{stress}}$ equals the integral of the density function f over the set $\{\Delta \boldsymbol{r} : f(\Delta \boldsymbol{r}) \leq f(\Delta \boldsymbol{r}_{\mathrm{stress}})\}$ which is one minus the integral of f over the set $\{\Delta \boldsymbol{r} : f(\Delta \boldsymbol{r}) > f(\Delta \boldsymbol{r}_{\mathrm{stress}})\}$.

In the Introduction we argued that stress tests are only useful if the scenarios have a certain minimal plausibility. Now that we have specified the

4

concept of plausibility we can translate the requirement of minimal plausibility into an admissibility domain for stress scenarios: given a certain plausibility threshold p between 0 and 1, the *admissibility domain* is the set of all scenarios with plausibility greater or equal to p.

3 Plausibility of Scenarios under Elliptical Distributions

In this section we calculate the plausibility of a given scenario $\boldsymbol{r}_{\text{stress}}$ and characterize the admissibility domain for a given plausibility threshold p, first for elliptical distributions in general and then for the special cases of the normal and the t-distribution.

Definition 2 *We call an n-dimensional distribution with density function f elliptical if f is of the form*

$$f(\Delta \boldsymbol{r}) = g(\Delta \boldsymbol{r}^T \cdot R \cdot \Delta \boldsymbol{r}), \qquad \Delta \boldsymbol{r} \in \mathbb{R}^n \tag{1}$$

where R is a symmetric, positive definite $n \times n$-matrix and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-negative function on the non-negative real numbers.

This definition is a special case of a more general definition which also applies to distributions without density (Fang *et al.* 1987). If g is a strictly decreasing function the distribution is unimodal and each surface of constant density is given by an ellipsoid. Furthermore, the admissibility domains will be the volumes contained in the ellipsoids of equal density.

Lemma 1 *If relative risk factor changes have an n-dimensional elliptical distribution with strictly decreasing g in representation (1), then the plausibility of a scenario $\boldsymbol{r}_{\text{stress}}$ is given by*

$$\text{Ell-Plaus}(\boldsymbol{r}_{\text{stress}}) = 1 - (\text{Det } R)^{-\frac{1}{2}} \frac{2 \pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_0^k t^{n-1} \, g(t^2) \, dt \tag{2}$$

with $k^2 = \Delta \boldsymbol{r}_{\text{stress}}^T \cdot R \cdot \Delta \boldsymbol{r}_{\text{stress}}$.

We write Ell-Plaus to indicate that we are now speaking about plausibility under the assumption of an elliptical distribution.

5

Proof The scenarios with higher probability density than $\boldsymbol{r}_{\mathrm{stress}}$ are the points contained in the ellipsoid

$$E(\boldsymbol{r}_{\mathrm{stress}}) = \{\Delta\boldsymbol{r} : \Delta\boldsymbol{r}^T \cdot R \cdot \Delta\boldsymbol{r} \le k^2\}.$$

Therefore, according to Definition 1, the plausibility of $\boldsymbol{r}_{\mathrm{stress}}$ is given by

$$\text{Ell-Plaus}(\boldsymbol{r}_{\mathrm{stress}}) = 1 - \int_{E(\boldsymbol{r}_{\mathrm{stress}})} g(\Delta\boldsymbol{r}^T \cdot R \cdot \Delta\boldsymbol{r})\, d(\Delta\boldsymbol{r}). \tag{3}$$

In order to evaluate the n-dimensional integral in (3), we introduce the function $h : \mathbb{R}^n \to \mathbb{R}^n$ by $h(\Delta\boldsymbol{s}) = A^{-1} \cdot \Delta\boldsymbol{s}$ where A is an $n \times n$-matrix with $R = A^T \cdot A$. A can be taken from the Cholesky decomposition of R. Since $R > 0$ we find $\text{Det}\, A \ne 0$ which assures the existence of A^{-1}. The rule of substitution leads to

$$\int_{E(\boldsymbol{r}_{\mathrm{stress}})} g(\Delta\boldsymbol{r}^T \cdot R \cdot \Delta\boldsymbol{r})\, d(\Delta\boldsymbol{r}) = \int_{h^{-1}(E(\boldsymbol{r}_{\mathrm{stress}}))} g\big(h(\Delta\boldsymbol{s})^T \cdot R \cdot h(\Delta\boldsymbol{s})\big)\, |\text{Det}\, h'(\Delta\boldsymbol{s})|\, d(\Delta\boldsymbol{s}).$$

One easily verifies $h^{-1}(E(\boldsymbol{r}_{\mathrm{stress}})) = S_n(k)$ and $|\text{Det}\, h'(\Delta\boldsymbol{s})| = (\text{Det}\, R)^{-1/2}$, where $S_n(k)$ is the n-dimensional sphere of radius k centered at the origin. This yields

$$\int_{E(\boldsymbol{r}_{\mathrm{stress}})} g(\Delta\boldsymbol{r}^T \cdot R \cdot \Delta\boldsymbol{r})\, d(\Delta\boldsymbol{r}) = (\text{Det}\, R)^{-\frac{1}{2}} \int_{S_n(k)} g(\Delta\boldsymbol{s}^T \cdot \Delta\boldsymbol{s})\, d(\Delta\boldsymbol{s}).$$

By introducing spherical coordinates one can verify the equation

$$\int_{S_n(k)} g(\boldsymbol{t}^T \cdot \boldsymbol{t})\, d\boldsymbol{t} = \int_0^k A(S_n(t))\, g(t^2)\, dt$$

where $A(S_n(t))$ is the surface area of $S_n(t)$. Applying (4) to the right-hand side of (4) yields the Lemma. $\qquad\square$

The admissibility domain for the plausibility threshold p is the set

$$\{\boldsymbol{r} : \Delta\boldsymbol{r}^T \cdot R \cdot \Delta\boldsymbol{r} \le k_p^2\}$$

where k_p is the k for which the right hand side of equation (2) equals p.

Multivariate normal distribution

Now we consider the special case where the n relative risk factor changes are normally distributed with mean zero and non-singular covariance matrix Σ. The density function is given by

$$f(\Delta \boldsymbol{r}) = \frac{(2\pi)^{-\frac{n}{2}}}{(\text{Det}\,\Sigma)^{\frac{1}{2}}}\, \exp\{-\frac{1}{2}\Delta \boldsymbol{r}^T \cdot \Sigma^{-1} \cdot \Delta \boldsymbol{r}\},$$

for $\Delta \boldsymbol{r} \in \mathbb{R}^n$. This is an elliptical distribution in the sense of Definition 2 with $R = \Sigma^{-1}$ and

$$g(t) = (2\pi)^{-\frac{n}{2}} (\text{Det}\,R)^{\frac{1}{2}}\, \exp(-\frac{t}{2}).$$

g is indeed strictly decreasing so that we can apply Lemma 1 to get the plausibility of a stress scenario:

$$\text{n-Plaus}(\boldsymbol{r}_{\text{stress}}) = 1 - \frac{2}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \int_0^k t^{n-1} \exp(-\frac{t^2}{2})\, dt\,.$$

Substituting $t^2 = x$ into the right hand side of this equation we get

$$\begin{aligned} \text{n-Plaus}(\boldsymbol{r}_{\text{stress}}) &= 1 - \frac{1}{2^{n/2}\,\Gamma\left(\frac{n}{2}\right)} \int_0^{k^2} x^{\frac{n}{2}-1} \exp(-\frac{x}{2})\, dx \\ &= 1 - F_{\chi_n^2}\left(k^2\right) \end{aligned} \tag{4}$$

where $k^2 = \Delta \boldsymbol{r}_{\text{stress}}^T \cdot \Sigma^{-1} \cdot \Delta \boldsymbol{r}_{\text{stress}}$ and $F_{\chi_n^2}(k^2)$ is the value of the χ^2-distribution function with n degrees of freedom at k^2. We write n-Plaus to indicate that we are now speaking about plausibility under the assumption of a normal distribution.

The admissibility domain for the plausibility threshold p is the set

$$\{\boldsymbol{r} : \Delta \boldsymbol{r}^T \cdot \Sigma^{-1} \cdot \Delta \boldsymbol{r} \leq k_p^2\}$$

where k_p^2 is given by the $(1-p)$-quantile of the χ^2-distribution with n degrees of freedom.

Multivariate t-distribution

When judging the plausibility of extreme market moves it is inappropriate to assume market changes are normally distributed. This gives unrealistically low plausibilities since financial data are usually fat-tailed. The class of t-distributions has been used with more success to describe financial time series, at least in one dimension. Here we use a multivariate t-distribution for calculating the plausibility of stress scenarios.

Multivariate t-distributions are n-dimensional distributions for which all marginals are t-distributions. The most commonly used is the *t-distribution with common denominator* (Johnson and Kotz 1972, 134ff). Its density function is

$$f(\boldsymbol{x}) = \frac{\Gamma(\frac{\nu+n}{2})(\operatorname{Det}\Lambda)^{-\frac{1}{2}}}{(\pi\nu)^{n/2}\,\Gamma(\frac{\nu}{2})}\,(1 + \frac{\boldsymbol{x}^T \cdot \Lambda^{-1} \cdot \boldsymbol{x}}{\nu})^{-\frac{\nu+n}{2}}, \tag{5}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, ν is a positive real number, and Λ is a positive definite, symmetric matrix. All the marginal distributions are t-distributions with the same number of degrees of freedom, namely ν. For $\nu > 2$ the covariance matrix exists and is given by $\frac{\nu}{\nu-2}\Lambda$.

In order to describe market moves with such a t-distribution $(\nu > 2)$we first re-scale risk factor changes so that they all have variance $\nu/(\nu - 2)$. The re-scaled relative risk factor changes are given by

$$\overline{\Delta\boldsymbol{r}} := (\sqrt{\frac{\nu}{\nu-2}}\,\frac{\Delta r_1}{\sigma_1}, \ldots, \sqrt{\frac{\nu}{\nu-2}}\,\frac{\Delta r_n}{\sigma_n})$$

where σ_i is the standard deviation of the i-th original risk factor change Δr_i. Now assume that the re-scaled relative changes $\overline{\Delta\boldsymbol{r}}$ are distributed according to the density (5). So the re-scaled risk factor changes have mean zero and covariance matrix $\frac{\nu}{\nu-2}\Lambda$. (Λ is the correlation matrix of the original risk factor changes $\Delta\boldsymbol{r}$.)

In order to calculate t_ν-Plaus($\boldsymbol{r}_{\text{stress}}$), the plausibility of a scenario $\boldsymbol{r}_{\text{stress}}$ under a multivariate t-distribution with common denominator and with ν degrees of freedom, we note that this is an elliptical distribution in the sense of Definition 2 with $R = \Lambda^{-1}$ and

$$g(t) = \frac{\Gamma(\frac{\nu+n}{2})}{(\pi\nu)^{n/2}\,\Gamma(\frac{\nu}{2})}\,(\operatorname{Det}R)^{\frac{1}{2}}\,(1 + \frac{t}{\nu})^{-\frac{\nu+n}{2}}\,.$$

By Lemma 1 we get

$$\text{t}_\nu\text{-Plaus}(\boldsymbol{r}_{\text{stress}}) = 1 - \frac{2\,\Gamma\!\left(\frac{\nu+n}{2}\right)}{\nu^{n/2}\,\Gamma\!\left(\frac{\nu}{2}\right)\Gamma\!\left(\frac{n}{2}\right)} \int\limits_0^j t^{n-1}\left(1+\frac{t^2}{\nu}\right)^{-\frac{\nu+n}{2}} dt \qquad (6)$$

with $j^2 = \overline{\Delta}\boldsymbol{r}^T_{\text{stress}} \cdot \Lambda^{-1} \cdot \overline{\Delta}\boldsymbol{r}_{\text{stress}} = \frac{\nu}{\nu-2}k^2$, where $k^2 = \Delta\boldsymbol{r}^T_{\text{stress}} \cdot \Sigma^{-1} \cdot \Delta\boldsymbol{r}_{\text{stress}}$ with Σ the covariance matrix of the original relative risk factor changes.

The admissibility domain for the plausibility threshold p is the set

$$\{\boldsymbol{r} : \overline{\Delta}\boldsymbol{r}^T \cdot \Lambda^{-1} \cdot \overline{\Delta}\boldsymbol{r} \leq j_p^2\}$$

where j_p is the j for which the right hand side of equation (6) equals p.

What makes t-distributions a popular choice for modeling market data is that they are elliptical and heavy tailed. Extreme events have a higher plausibility under the assumption of t-distributed changes of risk factors than under the assumption of normally distributed changes. The difference is enormous. Black Friday under the assumption of normality has a plausibility of $2 \cdot 10^{-77}$, which corresponds to a once in $2 \cdot 10^{74}$ years-event. In contrast, under the assumption of t_4-distributed changes, Black Friday has a plausibility of $2 \cdot 10^{-4}$, which corresponds to a once in 20 years-event.[1] Here is a table of the values of n-Plaus and of t_4-Plaus for $n = 5$, 50, and 500 dimensions and for $k = 5, 10$, and 15. k measures the size of the ellipsoid: the lengths of the major axes are proportional to k.

Dimensions	Size	t_4-Plaus	n-Plaus
n=5	k=5	0.0222	0.00014
	10	0.00165	$5.29 \cdot 10^{-20}$
	15	0.00034	$1.26 \cdot 10^{-46}$
n=50	k=5	0.5836	0.9988
	10	0.0917	$3.45 \cdot 10^{-5}$
	15	0.0219	$4.78 \cdot 10^{-24}$
n=500	k=5	$1 - 9 \cdot 10^{-8}$	$1 - 2 \cdot 10^{-224}$
	10	0.9582	$1 - 4 \cdot 10^{-90}$
	15	0.6495	$1 - 4 \cdot 10^{-29}$

---

[1]These numbers for the plausibility were calculated from equations (4) and (6) for a scenario representing the joint move of seven important risk factors on Black Friday, October 16, 1987. The covariances were estimated from the daily changes in the year 2000.

Two points are noteworthy. First, if we fix n, we observe the consequencies of the fact that the t_4-distribution is more fat-tailed than the normal distribution: For large moves the t_4-distribution has a higher density than the normal distribution, therefore the plausibility of large moves is higher under the t_4-distribution. Second, if we fix k, we observe that the plausibility of a move of a certain size k increases as the number n of dimensions increases. Or, put in other words, if we admit moves above a certain plausibility threshold, the size k of the admitted moves increases as the number of dimensions increases. This holds for both, the normal and the t_4-distribution. In the special case of linear portfolios, Studer (1997: p. 44) drew attention to a drawback of MaxLoss which results from this second observation.

We are aware that t-distributions still do not capture all of the properties of financial data. As compared to the normal distributions, they at least provide better modeling of fat tails of the marginal distributions and also have more realistic multivariate properties, as for example a non-zero tail dependence (Lindskog, 2000). Still, the multivariate t-distribution belongs to the class of elliptical distributions while financial data tend to be non-elliptically distributed.

Additionally, the assumption of changes being t-distributed is at odds with the Black-Scholes framework often assumed for the valuation of options. This threat of inconsistency materializes when on the one hand for defining the admissibility domain risk factor changes are assumed to be t-distributed and on the other hand for evaluating the loss of option positions risk factor changes are assumed to be normally distributed.

4   References

Artzner P, Delbaen F, Eber J-M, Heath D. 1999. Coherent Measures of Risk. *Mathematical Finance* **9** (3): 203-228. Accessible at http://www.math.ethz.ch/~delbaen/ [December 19, 2001]

Fang KT, Kotz S, Ng KW. 1987. *Symmetric Multivariate and Related Distributions*. Chapman & Hall: London.

Johnson NL, Kotz S. 1972. *Distributions in Statistics: Continuous Multivariate Distributions*. Wiley & Sons: New York.

Lindskog F. 2000. *Modelling Dependence with Copulas and Applications to*

Risk Management. Master Thesis, ETH Zurich. Accessible at http://www.risklab.ch/Papers.html [December 19, 2001]

Studer G. 1997. *Maximum Loss for Measurement of Market Risk.* Ph.D. Thesis No. 12397, ETH Zurich. Accessible at http://www.risklab.ch/Papers.html [December 19, 2001]

Studer G. 1999. Market risk computation for nonlinear portfolios. *Journal of Risk* **1** (4), 33-53.