Aggregation Issues in Operational Risk

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Abstract

In this paper we study copula-based models for aggregation of operational risk capital across business lines in a bank. A commonly used method of summation of the value-at-risk (VaR) measures, that relies on a hypothesis of full correlation of losses, becomes inappropriate in the presence of dependence between business lines and may lead to over-estimation of the capital charge. The problem can be further aggravated by the persistence of heavy tails in operational loss data; in some cases, the subadditivity property of value-at-risk may fail and the capital charge becomes under-estimated. We use α-stable heavy-tailed distributions to model the loss data and then apply the copula approach in which the marginal distributions are consolidated in the symmetric and skewed Student *t*-copula framework. In our empirical study, we compare VaR and conditional VaR estimates with those obtained under the full correlation assumption. Our results demonstrate significant reduction in capital when a *t*-copula is employed. However, the capital reduction is significantly smaller than in cases where a moderately heavy-tailed or thin-tailed distribution is calibrated to loss data. We also show that for confidence levels below 94% VaR exhibits the super-additivity property.

1. INTRODUCTION

The financial industry's attention to operational risk has been on increase in recent years. Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events.[1] Operational risk is believed to be largely a firm-specific non-systematic risk: According to the Basel Committee, "unlike market and perhaps credit risk, the [operational] risk factors are largely internal to the bank."[2]

In 2001, the Basel II Capital Accord (hereforth, Basel II) provided a detailed set of guidelines for banks on the basis of which they are required to estimate operational risk capital to serve as a buffer against potential future losses. The firm-specific nature of operational risk has prompted development of statistical models that make efficient use of historic operational loss data on the basis of which the capital charge can be estimated. The most sophisticated of such approaches, and the one most favoured by regulators[3] is the Loss Distribution Approach (LDA). LDA falls in the category of the Advanced Measurement Approaches.[4] LDA is the most accurate from the statistical point of view as it utilizes the exact distribution of historic losses – both frequency and severity – and is based on an individual bank's internal loss data. The core principle of the capital charge estimation under this approach is the value-at-risk (hereforth, VaR) metric that is measured based on a principle of aggregation of the frequency and severity distributions of losses forecasted for a one-year ahead time horizon.

Two tasks are central to an accurate estimation of operational VaR. The shape of the upper tail of the loss distribution largely determines the amount of the risk capital. One approach, extreme value theory (EVT), involves a separation of the main body of the loss distribution from the tails and modelling the tails with a Generalized Pareto Distribution (GPD); see for example, Chavez-Demoulin, Embrechts, and Neslehova (2006). In operational risk, EVT has been also used to model external data that is then used to "populate" scarce internal data; see Baud, Frachot, and Roncalli (2002) for a more detailed discussion.[5]

Aggregation of operational losses across business units or event types (or both) remains as important issue. A simplistic approach involves estimation of VaR measures for each cell independently and them adding them up to produce the aggregate measure of bank's risk. The problem with such an approach is that it assumes a perfect positive correlation between cells. As a result, the aggregate measure of risk represents an upper bound for a bank's true total level of risk. This property, often referred to as *sub-additivity* of risk measures, suggests that a bank can effectively reduce its risk capital by taking into account the dependence structure that exists between cells.

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[1] See BCBS (2001*b*).
[2] See BCBS (1998).
[3] Banks are allowed to choose an approach based on the bank's size and risk exposure and the ability to meet required criteria.
[4] A detailed description of the approaches to measure the operational risk capital charge is documented in BCBS (2001*a,b*; 2006).
[5] Overview of EVT and its applications to risk management can be found in Embrechts, Kluppelberg, and Mikosch (1997). Applications to operational risk are discussed in Chernobai, Rachev, and Fabozzi (2007) and McNeil, Frey, and Embrechts (2005), among others.

Some problems with sub-additivity are often cited in literature. The sub-additivity property may fail when risk is measured by VaR (see Artzner, Delbaen, Eber, and Heath (1999)). In this sense, VaR is not a coherent measure of risk. On the contrary, conditional value-at-risk (CVaR) is coherent in that the sub-additivity property holds. The failure of sub-additivity (i.e., super-additivity) may be aggravated in the presence of heavy tails in the loss data. As shown in Ibragimov and Walden (2007), this may have adverse implication for diversification. Our empirical analysis, presented later in this paper, also shows evidence of super-additivity.

In this paper, we extend discussion in Giacometti, Rachev, Chernobai, Bertocchi, and Consigli (2007) that focused on various modelling techniques for operational loss data. Here, we examine the issue of aggregation of operational losses across different business lines using a copula approach. Copulas are gaining popularity in measuring dependence in financial risk. They will be described later in this paper. In operational risk, copulas have been applied by Chavez-Demoulin, Embrechts, and Neslehova (2006), Embrechts and Puccetti (2006a,b), Dalla Valle, Fantazzini, and Giudici (2007), and Chapelle, Crama, Hubner, and Peters (2004), to name a few. Using copulas tends to lead to significant reductions in total VaR. For example, Chapelle, Crama, Hubner, and Peters (2004) showed that for a 99% VaR a bank can achieve a 35% reduction in capital estimates by using copula, echoing the results in Dalla Valle, Fantazzini, and Giudici (2007) who demonstrated that using copulas can result in savings for a bank in the range of 30% to 50%.

We use *t*-copulas to model the dependence between losses. *t*-copulas are optimal for modelling dependence between operational losses from different groups in that they succeed in capturing dependence in the tails. Other copulas, such as the Gaussian copula, has no tail dependence. For the loss distribution we use (i) a variation of the α-stable distribution and (ii) a mixture of the α-stable distribution and Generalized Pareto distribution. Such choice of loss distributions ensures that we do not leave out any extreme events and they are appropriately accounted for. We apply the methodology in an empirical study to operational loss data of a European bank.

The paper is organized as follows. Section 2 describes a statistical model for operational risk. Section 3 presents the data and empirical methodology. Section 4 describes the results of applying copulas to operational loss data and its implications for operational risk management. Finally, Section 5 concludes the paper and summarizes the findings.

2. A Modified Loss Distribution Approach for Operational Risk

The concept of compound Poisson process provides an accurate analytical framework to address the modelling problem of operational risk and is utilized in the Loss Distribution Approach (LDA) of the Advanced Measurement Approaches proposed by Basel II. The timing of the events is captured by the intensity of the Poisson process and the losses by an appropriate state distribution. Consider a bank with K "business lines/event type" combinations, $i=1,2,\ldots,K$. Then the aggregate operational loss for i-th business line are considered to follow a random process $\left\{L_t^i\right\}_{t\geq 0}$ with

$$L_t^i = \sum_{k=0}^{N_t^i} X_k^i, \quad X_k^i \propto F_\theta, N_t^i \propto Poi(\lambda). \tag{1}$$

The operational loss distribution is thus jointly determined by the average number of losses per unit of time – the intensity λ of the Poisson process N_t, the counting process with integer variables – and by the loss magnitudes X_k – in monetary terms – observed over a pre-determined interval of time usually taken to be one year. In our case X_k belong to a family F_θ of parametric continuous distributions.

Value-at-Risk is a risk measure that can be used as proxy for capital charge. It is computed as a high quantile (such as 99.9%) of the aggregate loss distribution. For cell *i*,

$$VaR_t^i = G_L^{-1}(0.999) \tag{2}$$

Where G is the cumulative distribution of L. Then, under the LDA, the total capital charge for a bank with M cells can be estimated as the sum of VaR measures[6] across cells:

$$K = \sum_{i-1}^{M} VaR_t^i \tag{3}$$

This LDA methodology assumes that the losses belonging to different "business line/event type" cells are perfectly positively correlated with each other. However, if the correlation is not perfect, under "nice" conditions LDA provides an upper bound for the total capital charge for a bank. As a result, LDA would generally tend to over-estimate the amount of risk capital a bank should use.

3. Data and Methodology

The data set selected for the study covers operational losses for a large European bank from January 2002 to August 2005 – a total of 3 years and 8 months. The sample consists of a total of slightly under 2,700 observations. The data are classified in accordance with the Basel II guidelines into business lines and event types. Due to relatively small data samples, the business lines considered for this study are: Retail Banking (using the Basel II definitions, this refers to business line 3 or BL3), Commercial Banking (BL4), and Retail Brokerage (BL8). We do not further granulate the data by event types for our estimations. BL3 accounts for 77.69% of the data, BL4 accounts for 7.82%, and BL8 accounts for 8.17% of our sample.

[6] An alternative to VaR is conditional VaR (or CVaR) which is estimated as the average loss given that it exceeds VaR.

Table 1. Descriptive statistics for internal, external, and pooled operational loss data (in Euro).

Business Line	Sample Statistic	Internal Data		External Data		Pooled Data	
		original	Log-scale	original	Log-scale	original	Log-scale
Retail Banking (BL3)	Mean	15,888.12	8.12	37,917.35	9.71	11,021.80	8.36
	Median	2,500	7.82	13,200	9.49	3,406.98	8.13
	St. Dev.	97,668.29	1.39	184,787.23	0.99	59,968.38	1.03
	Skewness	20.17	1.02	26.53	1.29	26.60	1.32
	Kurtosis	516.07	4.16	910.88	5.23	953.26	5.41
Commercial Banking (BL4)	Mean	28,682.11	8.11	61,880.06	9.73	10,175.51	7.59
	Median	2,235.60	7.71	12,080	9.40	1,347.44	7.21
	St. Dev.	133,873.92	1.67	653,409.10	1.11	83,252.98	1.24
	Skewness	7.60	1.13	29.65	1.44	19.11	1.53
	Kurtosis	63.14	4.01	904.11	5.70	413.87	6.02
Retail Brokerage (BL8)	Mean	13,089.05	8.11	40,808.30	9.60	19,674.84	8.87
	Median	2,755.90	7.92	11,000	9.31	5,273.24	8.57
	St. Dev.	48,801.21	1.40	305,524.49	0.99	144,912.26	0.99
	Skewness	9.49	0.89	35.67	1.59	35.87	1.58
	Kurtosis	108.83	3.62	1,642.25	6.81	1,668.56	6.71

We populate our sample with additional data extracted from an external database. The external database is a consortium-type database provided by a European vendor.[7] The data that constitute the database are collected from nearly 200 institutions. The size of the external data used for the analysis in this paper is approximately six times the sample size of internal data. In the external database, BL3 accounts for 60.63%, BL4 accounts for 6.46%, and BL8 accounts for 28.71% of the data.

To combine the internal data with external, we first standardized all data by the respective means and standard deviations and then scaled the pooled data back using the mean and the standard deviation of the original internal data to obtain an expanded internal dataset. Table 1 summarizes descriptive statistics of the data.

For the frequency distributions we considered a non-homogeneous Poisson process model; see Giacometti et al. (2007) for further details. For the loss distributions, we considered a large spectrum of candidate distributions: Lognormal, Generalized Pareto, Weibull, Logweibull, and symmetrised α-stable.

[7] We refrain from providing the consortium name in order to preserve confidentiality of the bank.

Symmetrised α-stable distribution belongs to the class of stable distributions.[8] For the symmetrised α-stable distribution we symmetrised the data by applying the transformation $Y = [X; -X]$ to the original data X and then fitting a 2-parameter (shape and dispersion) symmetric α-stable distribution. The advantage of fitting a symmetric α-stable distribution to symmetrised data over fitting a 4-parameter (shape, dispersion, skewness, and location) α-stable distribution to original data is that symmetric α-stable distribution requires estimation of only two parameters (shape and dispersion) allowing for more efficient estimates. The shape parameter α identifies the heaviness in the tail: $\alpha>1$ refers to a moderately heavy tail with a finite mean, while $\alpha\leq 1$ indicates a very heavy-tailed distribution with an infinite mean.

We refer as Model 1 to the approach in which a hypothesized loss distribution is fitted to the entire dataset. Goodness-of-fit test results showed that the symmetric α-stable distribution resulted in best fit to our full-sample loss data.[9] For the three business lines, BL3, BL4, and BL8, the shape parameters were estimated as 0.99, 0.83, and 0.99, respectively, and point to a very heavy tail.

Table 2. Estimates of the shape parameters.

Panel A: Full-sample distribution approach (Model 1)

	α of the symmetrised α-stable distribution
BL3	0.99
BL4	0.83
BL8	0.99

Panel B: Mixture distribution approach (Model 2)

	Body	**Tail**	
	α of the symmetrised α-stable distribution	**1/ξ of the Generalized Pareto distribution**	
		MLE estimate	**Hill estimate**
BL3	0.99	1.22	1.03
BL4	0.83	0.94	0.84
BL8	0.99	1.11	1.03

In a separate approach (Model 2), we combined Extreme Value Theory with the notion of mixture distributions to model the loss data. We first identified a level of threshold beyond which the losses were assumed to follow the Generalized Pareto Distribution (GPD). We then estimated the parameters of the GPD for the tails with the Maximum

[8] See Rachev and Mittnik (2000) for a thorough discussion.
[9] See Giacometti et al. (2007) for a detailed description of the goodness-of-fit test results.

Likelihood estimator (MLE) and the Hill estimator, and used symmetric α-stable distribution to model the body of the loss data. Finally, we combined the two distributions into a 2-model mixture distribution constructed as the weighted average of the two member distributions. Table 2 summarizes the estimated shape parameters from Models 1 and 2. The estimates suggest very heavy-tailed mixture distributions especially for BL4.

4. Aggregation of Losses Using Copulas

This section focuses on the discussion of two issues. The first issue is related to dependence between losses belonging to different "business line/event type" combinations.[10] When the correlation is sub-perfect, i.e. Equation 3 provides an upper bound for a bank's risk capital, the knowledge on the form of the dependence structure is needed. Computing correlation between cells implies a linear form of dependence and therefore may not be an optimal solution. If the dependence is of a non-linear nature, then copulas provide a natural solution.

A second issue we will address in this section is examining scenarios in which subadditivity property of VaR fails. Sub-additivity property dictates that

$$\sum_i VaR_i \geq VaR\left(\sum_i L_i\right) \tag{4}$$

Sub-additivity property may fail in cases when a loss distribution is very heavy-tailed. In essence, this means that the VaR of aggregate losses may actually exceed the sum of the VaR measures computed for each cell separately.

4.1. Definition of Copula

Definition 1 (Copula): *A copula function is a mapping from a set of univariate marginals to their full multivariate distribution. For m uniform random variables $U_1, U_2, ..., U_m$, the joint distribution function C, or copula, is*

$$C(u_1, u_2, ..., u_m, \rho) = P\left[U_1 \leq u_1, U_2 \leq u_2, ..., U_m \leq u_m\right]. \tag{5}$$

Copula functions can be used to link given marginal distributions with a joint distribution, since for given marginal distribution functions $F_1(l_1), F_2(l_2), ..., F_m(l_m)$, we have:

$$\begin{aligned} C(F_1(l_1), ..., F_m(l_m), \rho) &= P\left[U_1 \leq F_1(l_1), ..., U_m \leq F_m(l_m)\right] \\ &= P\left[F_1^{-1}(U_1) \leq l_1, ..., F_m^{-1}(U_m) \leq l_m\right] \\ &= P\left[L_1 \leq l_1, ..., L_m \leq l_m\right] = F(l_1, ..., l_m) \end{aligned} \tag{6}$$

[10] For the sake of simplicity and data consideration, in this paper we classify losses by business lines only.

Sklar (1959) established the fundamental converse result: he showed that any multivariate distribution function F can be written in the form of a copula function, namely: If $F_1(l_1, l_2, ..., l_m)$ is a joint multivariate distribution function with univariate marginal distribution functions $F_1(l_1), F_2(l_2), ..., F_m(l_m)$, then there exists a copula function $C(u_1, u_2, ..., u_m)$ such that $F_1(l_1, l_2, ..., l_m) = C(F_1(l_1), F_2(l_2), ..., F_m(l_m))$. If each F_i is continuous then the copula is unique.

Although copulas may be difficult to work with, the convenient aspect of copula estimation is that it can be performed independently from the estimation of the marginal severity distributions. In this paper, we consider symmetric Student *t*-copula and a skewed Student *t*-copula because these copulas are capable of capturing the dependence in the upper tail of the distributions.

4.2. Symmetric Student *t*-Copula

A symmetric Student *t*-copula $c(u_1, ..., u_m, R, \nu)$ is a function of the degrees of freedom ν and the correlation matrix R. For the 3-business line example, the symmetric *t*-copula has joint distribution function of the form:

$$c(u_1, u_2, u_3; R, \nu) = \frac{\Gamma((\nu+3)/2)[\Gamma(\nu/2)]^3 (1 + w^T R^{-1} w)^{-(\nu+3)/2}}{|R|^{0.5} \Gamma(\nu/2)[\Gamma(\nu+1/2)]^3 \Pi_{i=1,2,3}(1 + w_i^2/\nu)^{-(\nu+1)/2}} \tag{7}$$

where the vector $w = (w_1, w_2, w_2)^T, w_i = t_\nu^{-1}(u_i)$ has components directly computed from the inverse *t* distribution.

The degrees of freedom of the copula can be estimated using a recursive procedure proposed by Mashal and Naldi (2001). Given the initial estimate

$$R_0 = \frac{1}{T} \sum_{t=1,..,T} z_t z_t^T, \quad z_t = (z_t^1, z_t^2, z_t^3)^T, z_t^i = \Phi^{-1}(\tilde{l}_t^i), \tilde{l}_t^i = F(l_t^i) \tag{8}$$

the degrees of freedom in Equation 7 are determined as optimal values of the log-copula: For increasing $\nu_j = \nu_{j-1} + d\nu, \nu_0 = 1$, we generate recursively for $k = 0,1,2,...$

$$R_{k+1} = \frac{\nu_j + 3}{T\nu_j} \sum_{t=1}^{T} \frac{t_t t_t^T}{1 + \frac{1}{\nu_j} t_t^T R_k t_t} \qquad t_t := (t_t^1, t_t^2, t_t^3) = t_\nu^{-1}(\tilde{l}_t^1, \tilde{l}_t^2, \tilde{l}_t^3), \tilde{l}_t^i = F(l_t^i) \tag{9}$$

Then, a limit correlation matrix is input into the copula function up to the point in which the log-likelihood is maximised for the current correlation matrix and degrees of freedom:

$$\nu = \arg\max \sum_{t=1}^{T} \ln c_{\nu_j, \tilde{R}}^t (l_t^1, l_t^2, l_t^3, \nu_j \mid \tilde{R}), \tilde{R} = \lim_{k=1,2,...,} (R_k | \nu_j). \tag{10}$$

The iterative procedure will converge to a copula estimate that can be then incorporated into the simulator.

We estimated the limit value of the degrees of freedom to be between 6 and 7. The degrees of freedom and the optimal correlation matrix allow for a correct definition of the multivariate density for a bank's aggregate losses.

The empirical correlation and copula-based correlation structures can be illustrated in a 3-dimensional space. Figure 1 shows the joint losses of the three business lines' internal data on the top left and the empirical joint distribution on the top right. Simulated losses are on the bottom left and the symmetric *t*-copula-based joint distribution are in the bottom right corner. Comparison of the figures in the top row and in the bottom row suggests that the symmetric *t*-copula succeeds in effectively "reproducing" the true dependence structure between the losses.

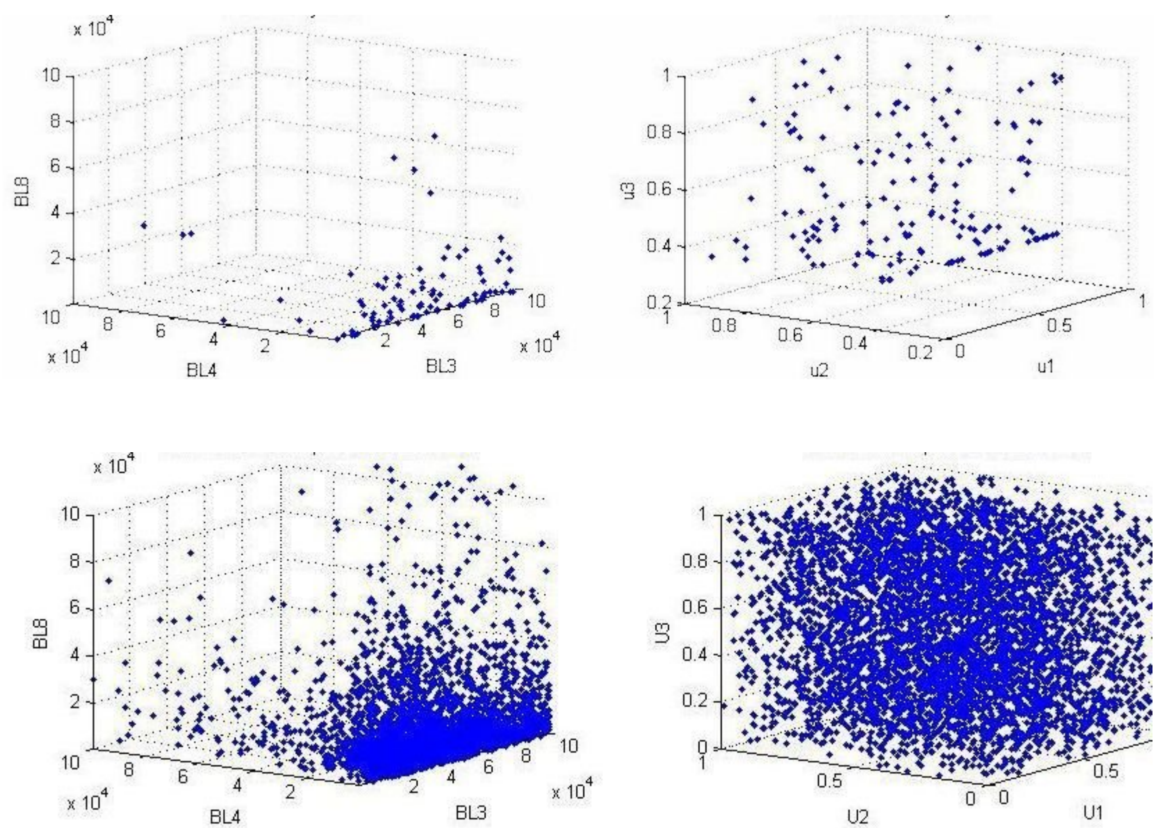

Figure 1. Correlation structure of the losses in the three business lines and a canonical example with a symmetric *t*-copula. *Top left:* Weekly aggregated losses, empirical data; *top right:* Empirical distribution of weekly aggregated losses; *bottom left:* Weekly aggregated losses, simulated data; and *bottom right:* Distribution of simulated losses, weekly aggregated using symmetric *t*-copula.

4.3. Skewed Student *t*-Copula

We use the following form of the multivariate skewed Student *t*-distribution for the copula function, defined by the stochastic representation as follows:[11]

$$X := \mu + \gamma W + Z\sqrt{W} \tag{11}$$

where $W \in IG(\nu/2, \nu/2)$ and $Z \in N(0,1)$, Z is independent of W, $\gamma = (\gamma_1, \ldots, \gamma_n)$ is an n-dimensional vector accounting for the skewness, $\mu = (\mu_1, \ldots, \mu_n)$ is n-dimensional location parameter vector, and ν is the degrees of freedom. We denote this distribution by $X \in t_n(\nu, \mu \Sigma, \gamma)$. The notation $IG(\nu/2, \nu/2)$ stands for the inverse Gamma distribution with both parameters equal to $\nu/2$. Thus, W is a one-dimensional random variable and Z is a random vector having a zero-mean multivariate normal distribution with covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \cdots \sigma_{1n} \\ \sigma_{21} & \sigma_{22} \cdots \sigma_{2n} \\ \vdots & \vdots \quad \vdots \\ \sigma_{n1} & \sigma_{n2} \cdots \sigma_{nn} \end{pmatrix}. \tag{12}$$

The multivariate skewed Student's *t*-distribution allows for closed-form expression of its density,

$$f_X(x) = \frac{aK_{(\nu+n)/2}\left(\sqrt{(\nu + (x-\mu)'\Sigma^{-1}(x-\mu))\gamma'\Sigma^{-1}\gamma}\right)\exp((x-\mu)'\Sigma^{-1}\gamma)}{(\nu + (x-\mu)'\Sigma^{-1}(x-\mu))\gamma'\Sigma^{-1}\gamma)^{-\frac{\nu+n}{4}}\left(1 + \frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu}\right)^{\frac{\nu+n}{n}}} \tag{13}$$

where $x \in R^n$, K is the modified Bessel function of the third kind and

$$a = \frac{2^{\frac{2-\nu-n}{2}}}{\Gamma(\nu/2)(\pi\nu)^{n/2}}\sqrt{|\Sigma|}. \tag{14}$$

The skewed Student's t copula is defined as the copula of the multivariate distribution of X. Therefore, the copula function is

$$C(u_1, u_2, ..., u_n) = F_X(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_{n1})) \tag{15}$$

[11] For more information, see Section 12.7 in Rachev and Mittnik (2000) and Demarta and McNeil (2005).

where F_X is the multivariate distribution function of X and $F_k^{-1}(u_k), k=1,\ldots,n$, is the inverse of the distribution function of the k-th marginal of X. That is, $F_X(x)$ has the density defined in Equation 13 and the density function $f_k(x)$ of each marginal is

$$f_k(x)=\frac{aK_{(\nu+n)/2}\left(\sqrt{(\nu+\frac{(x-\mu_k)^2}{\sigma_{kk}})\frac{\gamma_k^2}{\sigma_{kk}}}\right)\exp((x-\mu_k)\frac{\gamma_k}{\sigma_{kk}})}{((\nu+\frac{(x-\mu_k)^2}{\sigma_{kk}})\frac{\gamma_k^2}{\sigma_{kk}})^{-\frac{\nu+1}{4}}(1+\frac{(x-\mu_k)^2}{\nu\sigma_{kk}})^{\nu+1}},\quad x\in R \tag{16}$$

where σ_{kk} is the k-th diagonal element in the covariance matrix Σ. defined in Equation 12.

For the skewed Student's t-copula estimation we assume that there are n "business lines/event types" combinations of aggregate loss data samples: $X_{T\times n}=(X_1,X_2,\ldots,X_n)$.

We first estimate the parameters of the skewed Student's t-multivariate distribution on historical operational losses following the following procedure from Rachev, Stoyanov, and Milov (2007):

Step 1. Fit one-dimensional skewed Student's t-distribution over all risk variables on a stand-alone basis. The result from that step is:

$$(5,\hat{\mu}_i,\hat{\sigma}_i,\hat{\gamma}_i), i=1,\ldots,n \tag{17}$$

We use MLE method to obtain the estimates. For the degrees of freedom we use γ=5 because at this value the copula is most sensitive to the asymmetry of parameters; the skewed distribution reduces to the symmetric case at $\gamma=0$.

Step 2. Estimate the correlation matrix Σ of the multivariate skewed Student's t distribution by the following formula:

$$\hat{\Sigma}=(\mathrm{cov}(X)-\frac{2\nu^2}{(\nu-2)^2(\nu-4)}\hat{\gamma}\hat{\gamma}')\frac{\nu-2}{2} \tag{18}$$

where $\nu=5$ and $\hat{\gamma}=(\hat{\gamma}_1,\hat{\gamma}_2,\ldots,\hat{\gamma}_n)$.

Step 3. Adjust the matrix $\hat{\Sigma}$ so that it becomes positive definite.

Having estimated the parameters of the skewed t-distribution, we obtain the skewed t-copula using the following 2-step simulation algorithm:

Algorithm Step 1. Draw N independent n-dimensional vectors from the multivariate skewed Student's t distribution using the stochastic representation defined in Equation 11 and the set of fitted parameters $(5,\hat{\mu}_i,\hat{\sigma}_i,\hat{\gamma}_i), i=1,\ldots,n$. The result from that step is $N\times n$ matrix $S=\lfloor s_{ij}\rfloor$ with simulations. This is obtained by, first, drawing N independent n-dimensional vectors from the multivariate Normal distribution $N(0,1)$, second, drawing N independent random numbers from the inverse Gamma distribution with parameters $IG(\nu/2, \nu/2)$, and, third, obtaining final simulations using Equation 11 with the estimated parameter values.

Algorithm Step 2. Transform simulations S to uniform simulations U using the sample distribution function of the marginals. Denote by $\hat{F}_k(x)$ the sample cumulative distribution function of the *k*-th marginal,

$$\hat{F}_k(x) = \frac{1}{N}\sum_{j=1}^{N} I\{S_{jk} \le x\}, \tag{19}$$

where $I\{A\}$ stands for the indicator function of the set A. Then

$$U_{jk} = \hat{F}_k(S_{jk}),\ j = 1,\ldots,N \quad k = 1,\ldots,n. \tag{20}$$

4.4. Results of Copula Estimation with Operational Loss Data

In this section, we apply the symmetric Student *t*-copula and skewed Student *t*-copula to our data samples. We consider two models for the marginal distributions of the losses that were summarized in Section 3.

Table 3 summarizes the estimates of population descriptive statistics and risk capital measures at confidence levels 97.5, 98, 99, and 99.9 percent, by business line. These statistics refer to the marginals that will later be aggregated with copulas. It is notable that Model 2 produces uniformly significantly larger estimates than Model 1.

In order to estimate concordance measures between the three business lines, we aggregated losses on a weekly basis. Table 3 presents results of the Spearman correlation coefficient and Kendall correlation coefficient for each pair of the business lines. The estimates suggest very low degree of dependence between the losses.

To aggregate the losses belonging to the three business lines, we use the symmetric Student *t*-copula and skewed Student *t*-copula, described in Sections 4.2 and 4.3, respectively. The degrees of freedom were estimated for the former copula as ν=6.27 and for the latter copula as ν=5.

To apply the copulas to the loss data, we first estimate the copula parameters using historical losses aggregated weekly. We then repeat the following steps a very large number of times. In the first step, from the estimated copula we sample a multivariate random vector with marginals distributed as uniform [0,1] random variables. In the next step, for each business line, we obtain scenarios for the cumulative loss realization by inverting the uniform [0,1] variate from the previous step and then sum them up to obtain the total loss for the bank. Repeating these steps produces the distribution of aggregate losses. Finally, VaR and CVaR are computed from the obtained aggregate distributions.

Table 3. Population descriptive statistics and risk capital estimates by business line.

Panel A: Model 1.

	Population descriptive statistics (Euro '000 000)		Risk capital estimates (Euro '000 000)							
			VaR				CVaR			
	Mean	**St.Dev.**	**97.5%**	**98%**	**99%**	**99.9%**	**97.5%**	**98%**	**99%**	**99.9%**
BL3	0.022	0.239	0.070	0.092	0.211	4.271	0.605	0.736	1.337	6.376
BL4	0.008	0.048	0.051	0.059	0.106	0.854	0.190	0.223	0.370	1.043
BL8	0.026	0.141	0.175	0.197	0.353	1.914	0.507	0.586	0.920	3.377

Panel B: Model 2.

	Population descriptive statistics (Euro '000 000)		Risk capital estimates (Euro '000 000)							
			VaR				CVaR			
	Mean	**St.Dev.**	**97.5%**	**98%**	**99%**	**99.9%**	**97.5%**	**98%**	**99%**	**99.9%**
BL3	0.030	0.320	0.130	0.170	0.290	6.090	0.860	1.040	1.860	8.430
BL4	0.050	0.880	0.180	0.220	0.490	3.920	1.430	1.730	3.170	22.960
BL8	0.080	0.360	0.550	0.610	1.190	4.280	1.570	1.820	2.760	8.330

Table 4. Estimates of concordance.

		BL3	**BL4**	**BL8**
BL3	Spearman	1.000		
	Kendall	1.000		
BL4	Spearman	-0.004	1.000	
	Kendall	0.003	1.000	
BL8	Spearman	-0.040	0.026	1.000
	Kendall	-0.031	0.020	1.000

Table 5. Population descriptive statistics and risk capital estimates for aggregate loss data. The numbers in parentheses indicate percentage reduction (if "-") or increase (if "+") from the corresponding risk measure under perfect positive correlation scenario.

Panel A: Model 1.

	Population descriptive statistics (Euro '000 000)		Risk capital estimates (Euro '000 000)							
			VaR				CVaR			
	EL	**UL**	**97.5%**	**98%**	**99%**	**99.9%**	**97.5%**	**98%**	**99%**	**99.9%**
Perfect positive correlation	0.056	0.280	0.296	0.347	0.670	7.039	1.546	1.302	2.626	10.796
Symmetric t-copula	0.055 (-1.8%)	0.277 (-1.1%)	0.287 (-3.0%)	0.366 (-5.5%)	0.657 (-1.9%)	4.940 (-29.8%)	1.040 (-32.7%)	1.219 (-6.4%)	1.967 (-25.1%)	6.647 (-38.4%)
Skewed t-copula	0.055 (-1.8%)	0.298 (+6.4%)	0.281 (-5.4%)	0.351 (+1.2%)	0.645 (-3.7%)	5.228 (-25.7%)	1.065 (-31.1%)	1.254 (-3.7%)	2.051 (-21.9%)	7.106 (-34.2%)

Panel B: Model 2.

	Population descriptive statistics (Euro '000 000)		Risk capital estimates (Euro '000 000)							
			VaR				CVaR			
	EL	**UL**	**97.5%**	**98%**	**99%**	**99.9%**	**97.5%**	**98%**	**99%**	**99.9%**
Perfect positive correlation	0.155	1.004	0.859	0.990	1.963	14.289	4.591	3.860	7.782	39.721
Symmetric t-copula	0.152 (-1.9%)	0.978 (-2.6%)	0.853 (-0.7%)	1.021 (+3.1%)	1.936 (-1.4%)	11.311 (-20.8%)	3.135 (-31.7%)	3.680 (-4.7%)	5.995 (-23.0%)	24.877 (-37.4%)
Skewed t-copula	0.148 (-4.5%)	1.004 (0.0%)	0.835 (-2.8%)	1.020 (+3.0%)	1.857 (-5.4%)	10.277 (-28.1%)	3.064 (-33.3%)	3.596 (-6.8%)	5.870 (-24.6%)	25.761 (-35.1%)

Panel C: Historical data.

	Population descriptive statistics (Euro '000 000)		Risk capital estimates (Euro '000 000)							
			VaR				CVaR			
	EL	**UL**	**97.5%**	**98%**	**99%**	**99.9%**	**97.5%**	**98%**	**99%**	**99.9%**
Historical data	0.245	0.561	1.025	1.236	3.306	5.850	3.589	2.978	4.671	5.850
Historical data excluding worst 2 losses	0.209	0.340	0.868	0.977	1.404	3.492	2.021	1.739	2.459	3.492

Table 5 summarizes the estimates for expected and unexpected losses (EL and UL) and the bank's cumulative risk capital, derived from the copula-based approach and compared to the perfect correlation approach. The estimates are based on a one week horizon. It is notable that in our results there is a negligible difference in EL and UL and the effect is more pronounced for the VaR and CVaR estimates. In vast majority of cases copula approach results in a substantial reduction in risk capital for the bank. For example, for 99.9% VaR, reduction in capital ranges from 21% to 29% and for 99.9% CVaR, reduction in capital is roughly in the 34%-38% range. Another notable result is that if Model 1 is used, symmetric Student *t*-copula generally produces higher reduction in risk capital than the skewed Student *t*-copula, while the relation is reverse in most part if Model 2 is used. However, because skewed Student t-copula is a generalized version of the symmetric Student *t*-copula, a risk manager would prefer the former one.

Panel C of Table 5 shows capital estimates based on historical data. Comparison of the figures with those under Model 1 and Model 2 reveals that for percentiles below 99.9, the former model uniformly under-estimates the true historic loss while the latter model produces results fairly consistent (only slightly higher) with those actually experienced. Then, because Model 2 corresponds to a more heavy-tailed distribution than Model 1, the CVaR figures under Model 2 reveal overestimation of the historic counterparts roughly by a factor of 2. On the contrary, Model 1 in most part underestimates CVaR. A risk manager would favour Model 2 over Model 1 based on the above discussion.

Our findings indicate lower reduction in risk capital from those reported by Chapelle, Crama, Hubner, and Peters (2004) and Dalla Valle, Fantazzini, and Giudici (2007). Chapelle, Crama, Hubner, and Peters (2004) showed that for a 99% VaR a bank can achieve reduction in capital estimates by 35% by using copula, and Dalla Valle, Fantazzini, and Giudici (2007) showed reduction in the range of 30%-50%. One explanation is that we used a much more heavy-tailed loss distribution than in the two other studies. This means that, if a thin-tailed loss distribution is used to model loss data, copula-based aggregation would result in much more significant reduction in total capital. In essence, this implies that if a bank has mistakenly chosen to use a thin-tailed loss distribution when the data are in fact heavy-tailed, the resulting capital would be understated. The converse is also true. The choice of the loss distribution thus becomes of central concern and must be estimated with high degree of accuracy.

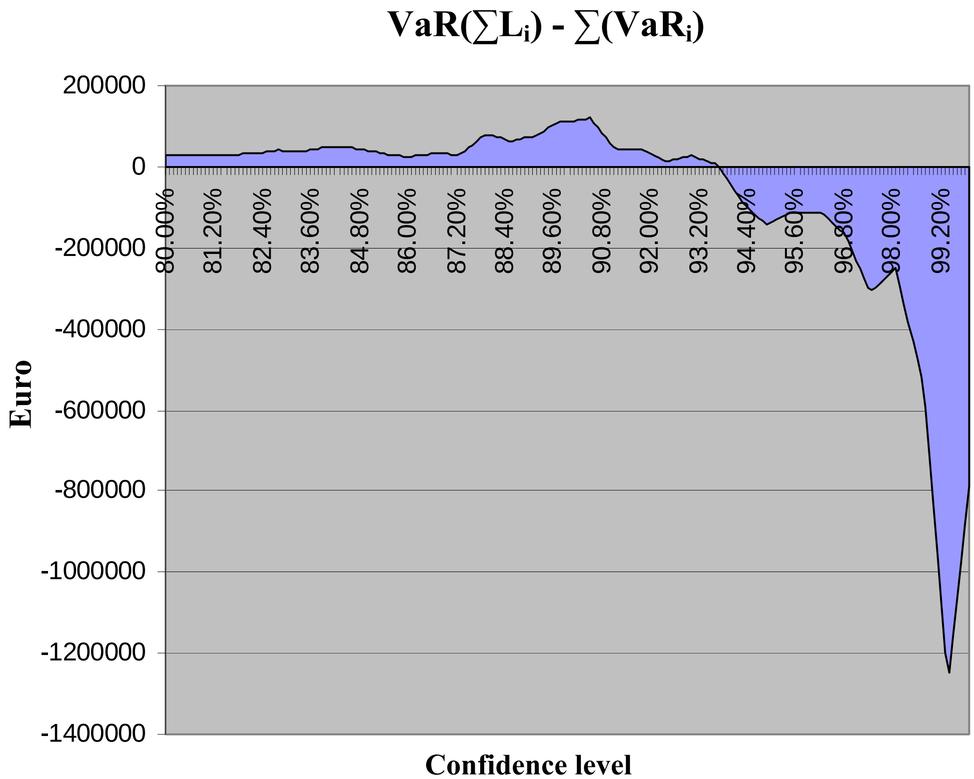

Figure 2. Illustration of super-additivity and sub-additivity of historic VaR.

4.5. Super-Additivity of VaR

Panels A and B of Table 5 show that for the 98% confidence level, we observe super-additivity of VaR measures in 3 out of 4 cases. Super-additivity is also observed in our results for the historic estimates of VaR under 99.9% for Models 1 and 2 and CVaR estimates under 99.9% for Model 1. Super-additivity has been documented in literature; see, for example, McNeil, Frey, and Embrechts (2005). Figure 2 illustrates super-additivity and sub-additivity in historical VaR estimates. The horizontal line is the benchmark representing a VaR estimate based on the aggregation of historical data for BL3, BL4, and BL8 using historical correlations. The plot represents the summation of individual VaR estimates for each of the three business lines. Super-additivity is observed for confidence levels under 94%. One must pay extra care when choosing a confidence level to provide VaR and CVaR measures: for confidence levels chosen too low, the estimated capital charge may be under-estimated while for confidence levels too high, it may be over-estimated. The phenomenon of super and sub-additivity seems more severe for more heavy-tailed loss distributions.

5. Conclusions

In this empirical paper we used a copula approach to model the dependence between operational losses belonging to different business lines. The contributions of this paper can be summarized as follows:

(1) For the loss distributions that constitute the marginals of copulas we used very heavy-tailed symmetric α-stable distribution (a member of α-stable distributions) and a combination of a symmetric α-stable distribution with the Generalized Pareto distribution for the tails. In operational risk, such heavy-tailed distributions are superior to thin-tailed and moderately heavy-tailed distribution due to their capacity to detect and account for the heaviness in the upper tails.

(2) We selected symmetric Student *t*-copula and skewed Student *t*-copula due to their ability to capture tail dependence. Our empirical study has shown that using *t*-copulas results in substantial reduction in risk capital for the bank. Although our findings are consistent with other empirical studies that have documented capital reduction when using copula, upon drawing comparisons with other similar studies, we have found that the magnitude in capital reduction is smaller than if thin-tailed or moderately-tailed distributions, such as Lognormal and Gamma, are used to constitute the marginals.

(3) Another finding was presence of the super-additivity phenomenon in the VaR estimates. For our data sample, super-additivity was observed for confidence levels below 94%.

Our findings effectively demonstrate that falsely settling on a loss distribution that is not sufficiently heavy-tailed for given data puts the capital estimates at the risk of severe underestimation for very high confidence levels or overestimation for confidence levels insufficiently high. Careful analysis and a variety of goodness-of-fit tests are thus crucial in selecting a loss distribution which is central to the accuracy of the capital charge estimates.

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