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## Goodness-of-Fit Test focuses on Conditional Value at Risk: An Empirical Analysis of Exchange Rates

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### Abstract.

To verify whether an empirical distribution has a specific theoretical distribution, several tests have been used, for example: Kolmogorov-Smirnov and Kuiper. These tests try to analyze if all parts of the empirical distribution has a specific theoretical shape. But, in a Risk Management framework, the focus of analysis is on the tails of the distributions, since we are interested on the extreme returns of financial assets. This paper proposes a new goodness-of-fit hypothesis test with focus on the tails of the distribution. The new test is based on the Conditional Value at Risk measure. Three major exchange rates (JPY/USD, GBP/USD and CHF/USD) are used as examples of a practical application of the test proposed. The new test, the Kolmogorov-Smirnov and Kuiper tests were applied to verify if the empirical data has a Normal, Scaled-t, Hyperbolic, NIG or GH distribution. For JPY currency, the Normal, Hyperbolic and scaled-t distributions were rejected by the new test. For the CHF and GBP, only Normality was rejected. Results are the same for CHF and GBP when using the other two tests. But for the JPY, the Scaled-t and the Hyperbolic distributions are rejected on the new test, and not rejected for the other two tests. We conclude that, for overall finance applications, we can use Scaled-t and Hyperbolic distributions for the JPY, but for Risk Management applications, they are not adequate.

**1. Introduction.** The PhD thesis of Louis Bachelier in 1900 (see Courtault et al [2000]), was the first to propose the use of a Normal distribution to model price movements in financial markets. On the Bachelier thesis, the absolute price increments of financial assets were modeled by a Normal distribution. But this model allows negative prices, so Osborne (1959) and Samuelson (1965) suggested the geometric Brownian motion, where the price percentage returns follow a Normal distribution. This assumption of Normality is widely used in risk models, like the Riskmetrics<sup>TM</sup>(1995, 1996), considered a benchmark model on measuring market risk.

Nevertheless, the Normality assumption was later refuted by several empirical researches, starting with Mandelbrot (1963) and Fama (1965). The actual distribution of financial assets has some differences from the normal distribution, as fat tails (positive excess of kurtosis) and asymmetry (see Rydberg [1997] and Hull and White [1998]). Then researches tried to find which other distributions could replace the Normal.

In the 1960's, Mandelbrot (1963) and Fama (1965) were the pioneers in proposing alternative distributions to model financial data. They proposed that the Stable Paretian distribution. In the 1970's, Praetz (1972) and Blattberg and Gonedes (1974), proposed the scaled Student  $t$  distribution, and in the 1980's, Kon (1984) suggested the mixture of Normals as an alternative to model financial assets. In

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the 1990's several papers analyzed the use of the Generalized Hyperbolic distribution to model financial assets, starting with Eberlein and Keller (1995).

When choosing among different distributions to model financial assets, a common problem that arises is to verify whether an empirical distribution has a specific distribution or not. Berkowitz (2002) suggests the distributional test as one of the steps to assess a risk model. Several tests have been used, for example: Kolmogorov-Smirnov, Likelihood Ratio and Kuiper. These tests try to analyze if all parts of the empirical distribution has a Normal shape. But, in a Risk Management framework, the focus of analysis is on the tails of the distributions, since we are interested on the extreme returns of financial assets. So, in a Risk Management framework, the goal of a goodness-of-fit test should be whether the tails of the theoretical distribution are a good approximation to the tails of the empirical distribution.

In this way usual goodness-of-fit tests would assess the goodness-of-fit of a theoretical distribution based on mismatch of all distribution, and not only the tails. Therefore, the main goal of this paper is to propose a new goodness-of-fit hypothesis test with focus on the tails of the distribution, i.e., a tail-goodness-of-fit test with focus on Risk Management. First, we propose a statistical distance based on the *Conditional Value at Risk* (CVaR)<sup>1</sup> of two distributions. Then we use Monte Carlo Simulation to perform hypothesis tests based on the distance proposed, using data from three major exchange rates. We test the hypothesis that the empirical distribution has a Normal, Scaled Student-t, Generalized Hyperbolic (GH), Normal Inverse Gaussian (NIG) and Hyperbolic distributions, based on the new distance proposed on this paper.

This paper is organized as follows: section two gives a brief overview of market risk models. Section three reviews some goodness-of-fit tests. On section four we introduce the new goodness-of-fit test proposed by this paper. Section five performs an empirical application of the new test, as well as other tests. Section six concludes the paper with the final remarks and suggestions for further research.

## 2. Market Risk Models.

**2.1. Market Risk Measures.** In recent years, risk management became popular among researchers, market practitioners and regulators. The Value at Risk (VaR) emerged as one of the benchmark measure for market risk.

According to Basak and Shapiro (2001), “*evidence abounds that in practice VaR estimates not only serve as summary statistic for decision makers, but are also used as a tool to manage and control risk – where economic agents struggle to maintain the VaR of their market exposure at a prespecified level*”.

The VaR allows the market risk to be expressed in one number: the loss one expected to suffer with a certain confidence level to a fixed holding period:

$$P[R < -VaR(\alpha)] = 1 - \alpha$$

Where  $R$  is the random variable of the asset's returns, and  $\alpha$  is the confidence level with which the VaR is being calculated.

The distribution of  $R$  can be an empirical non-continuous distribution, or a theoretical specified continuous distribution. The Value at Risk of an empirical distribution can also be viewed as the  $\alpha$  quantile of the distribution.

Although the intense use of VaR, researchers have criticized this risk measure. One question not addressed by the concept of VaR is what is the magnitude of the loss when the VaR limit is exceeded. Another issue on VaR pointed out by the article of Artzner, Delbaen, Eber and Heath (1999) is that

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<sup>1</sup>See section 2 for discussion about risk measures and Conditional Value at Risk.

it is not a “coherent” measure of risk. They say that a risk measure is a coherent measure of risk if it satisfies the following four properties:

- Translation invariance: for a constant  $k$ ,  $\rho(x + k) = \rho(x) + k$
- Subadditivity: for all  $X$  and  $Y$ ,  $\rho(X + Y) \leq \rho(X) + \rho(Y)$
- Positive homogeneity: for a positive constant  $k$ ,  $\rho(kx) = k\rho(x)$
- Monotonicity: For all  $X \leq Y$  for each outcome, then  $\rho(X) \leq \rho(Y)$

The VaR is not considered a coherent measure of risk, because it fails to hold the subadditivity property, i.e., the VaR of a two assets portfolio can be greater than the sum of the two individual VaR’s.

Also, Prause (1999) argues that to avoid bankruptcy one should forecast the distribution of the maximum expected loss. From this point of view regulators should use other risk measures than VaR. A better incorporation of extreme events especially in view of nonlinear portfolios is desirable.

Therefore, a new measure of risk has emerged in the literature, the Conditional VaR<sup>2</sup>, that is the expected value of the loss, given that it exceeds a certain level. The Conditional VaR (CVaR) can be written as:

$$CVaR(R, \alpha) = -E[R | R \leq -VaR(\alpha)]$$

Considering that distribution of  $R$  is known, we have:

$$CVaR(R, \alpha) = \frac{\int_{-\infty}^{-VaR(\alpha)} x f_R(x) dx}{1 - \alpha}$$

Where  $f_R$  is the probability density function of  $R$ .

This measure, differently from VaR, satisfies the subadditivity property above mentioned, and addresses a question not answered by VaR: “How bad is bad?”. The VaR informs only if the loss is above a certain level, but no information about the magnitude of the loss is given. So, the CVaR is used to answer this question.

**2.2. The Parametric Approach.** Several approaches can be used to estimate the market risk measures. The choice will depend on the kind of portfolio, computational resources available and time constraints. The main three approaches are: parametric, historical simulation and Monte-Carlo simulation. This paper focuses on the parametric approach.

The parametric approach assumes that the asset returns have a specific probability distribution, (for example, the Normal). The parameters of the distribution are estimated (for example, the volatility) and so the risk measure is calculated based on the estimated distribution. So the risk measure depends on the parameters used. The Normal distribution is the most used with this approach. The RiskMetrics? model is the most popular market risk model, and used this approach, assuming that asset returns follow a Normal distribution, with mean equal to zero, and volatility estimated by the EWMA (Exponential Weighted Moving Average) method, using historical data. Various other methods to estimate the volatility have been tested to replace the EWMA, usually methods of the GARCH family. This approach is usually called Conditional Normal methods, since we use a Normal distribution with the variance being calculated by conditional methods.

As the Normal distribution has thinner tails than the empirical distributions, other distributions different from the Normal have also been tested, such as student-t, mixture of Normals, Hyperbolic, etc.

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<sup>2</sup>This measure has been used with many other names such as Expected Shortfall and Tail VaR.

Hull and White (1998) used a mixture of two Normal distributions to assess the Value at Risk of 12 exchange rates from 1988 to 1997. The first half of data was used to estimate the parameters and the second to assess the model. An EWMA was used to estimate the volatility. Results when each currency has a separate estimation show the single currency model could be rejected for only four of the currencies, using the chi-squared test with 95% confidence. When the same parameters are used for all currencies, the model cannot be rejected with 95% confidence.

Bauer (2000) used a symmetric Hyperbolic distribution to perform VaR calculations, and used data from German stocks and international stock indexes (DAX, Dow Jones and Nikkei) from 1987 to 1997. His results showed that the model with Hyperbolic distribution outperformed Riskmetrics model.

Prause (1999) used the Generalized Hyperbolic distribution and its subclasses to fit German stocks, U.S. stock indexes and exchange rates. He used several statistics to assess the goodness-of-fit such as the Kolmogorov and Anderson-Darling distances, with the Normal being always the worst one. Also, Prause performed VaR calculations with single assets and multi-assets portfolios.

Also Generalized Pareto distributions are good candidates to fit extreme event data, then we can use these distributions to calculate VaR, for more details see Embrechts, Klueppelberg and Mikosch (1997).

The main advantage of the parametric approach is the speed of calculation. Also, when using some conditional volatility estimation like EWMA or GARCH (conditional parametric methods), this method captures better the market stresses situations. On the other hand, this method has limitations if applied to portfolios with non-linear instruments, such as options.

**3. Goodness-of-fit Tests.** To test whether an empirical distribution has a specific distribution (or not), several tests have been used. The most common is the Kolmogorov-Smirnov. The Kolmogorov distance (see, for example, Massey[1951]) is defined as the greatest distance between empirical distribution and theoretical distribution, for all possible values:

$$D_{Kol} = \max_{x \in \mathbb{R}} |F_{Emp}(x) - F_{Theo}(x)|$$

where  $F_{Emp}$  is the empirical cumulative function and  $F_{Theo}$  is the continuous and completely specified theoretical cumulative function.  $F_{Emp}$  can be defined by:

$$F_{Emp}(x) = (\text{number of } X_i' \leq x) / n$$

where  $X_i'$ s are the sample's elements and  $n$  is the sample number of elements.

The Kuiper (1962) distance is similar to the Kolmogorov distance, but considers both directions of the discrepancy adding the greatest distances upwards and downwards:

$$D_{Kui} = \max_{x \in \mathbb{R}} \{F_{Emp}(x) - F_{Theo}(x)\} + \max_{x \in \mathbb{R}} \{F_{Theo}(x) - F_{Emp}(x)\}$$

These tests try to analyze if all parts of the empirical distribution has a Normal shape. But, in a Risk Management framework, the analysis focus is on the tails of the distributions, since we are interested on the extreme returns of financial assets.

One approach to give emphasis on the tails of a distribution is to use the Anderson and Darling (AD) distance, proposed in a 1952's paper. They propose a distance that would be viewed as Kolmogorov distance with weight. Weighting can be defined giving special importance to tails, and so being especially relevant to risk measures. The formula of this distance with tail emphasis is:

$$D_{AD} = \max_{x \in \mathbb{R}} \frac{|F_{Emp}(x) - F_{Theo}(x)|}{\sqrt{F_{Theo}(x)[1 - F_{Theo}(x)]}}$$

Prause (1999) uses the AD distance to assess which theoretical distribution fits better the data of German Stocks. The distributions assessed were Normal, Generalized Hyperbolic, Hyperbolic and Normal Inverse Gaussian. Nevertheless, he did not perform a hypothesis test, he just compared the AD distance of the distributions, to find out which one is the best. According to Prause, the Normal distribution is the worst one to his set of data.

There are other kinds of distances and tests that give emphasis on the tail, like the ones proposed in Crnkovic and Drachman (1996) and Farias, Ornelas and Fajardo (2003)<sup>3</sup>. Both use distances similar to the Kuiper, but with greater weights on the tails. The FOF distance uses the weight function of the AD distance.

**4. Tail-Goodness-of-fit Test Considering CVaR.** In this paper, we propose a new distance to test the goodness-of-fit of a theoretical distribution to an empirical distribution.

The distance proposed here gives emphasis on the tails of the distribution (like the AD, CD and FOF tests) and so is more adequate to risk measures than the Kolmogorov distance. Our distance gives special emphasis on a specific risk measure – the Conditional Value at Risk (CVaR), and it is the absolute difference between the empirical and the theoretical CVaR for a given significance level. Intuitively, we are calculating the error of the theoretical distribution in estimating the CVaR of the empirical distribution. Mathematically:

$$\begin{aligned} D_{CV}(X, \alpha) &= |CVaR_{Theo}(X, \alpha) - CVaR_{Emp}(X, \alpha)| \\ &= \frac{1}{1 - \alpha} \left| \left( \int_{-\infty}^{VaR_{Emp}} x f_{Emp}(x) dx \right) - \left( \int_{-\infty}^{VaR_{Theo}} x f_{Theo}(x) dx \right) \right| \end{aligned}$$

Where:

- $X$  is the random variable representing the returns of the asset;
- $VaR_{Emp}$  is the Value at Risk calculated using the Empirical distribution with a confidence level  $\alpha$ ;
- $VaR_{Theo}$  is the Value at Risk calculated using the Theoretical with a confidence level  $\alpha$ ;
- $f_{emp}$  is the probability density function of the empirical distribution and
- $f_{theo}$  is the probability density function of the theoretical distribution.

Differently from the Kolmogorov, AD, CD, FOF and other distances analyzed on this paper, the distance proposed is not based on a Maximum operator over the cumulative probability distribution, but it is a sum over the density probability functions. This has the purpose to fit in a better way the CVaR measure, which is also a sum.

Note that the distance has a parameter  $\alpha$  besides the two distributions (empirical and theoretical). This is the level of confidence of the VaR beyond which the distance works. So, if one is interested on both tails of the distribution<sup>4</sup>, it is necessary to define a bi-caudal distance BCV (Bi-caudal Conditional Value at Risk):

$$D_{BCV}(X, \alpha) = |CVaR_{Theo}(X, \alpha) - CVaR_{Emp}(X, \alpha)| + |CVaR_{Theo}(-X, \alpha) - CVaR_{Emp}(-X, \alpha)|$$

Based on this distance, we can perform a hypothesis test with the null hypothesis that the empirical distribution is equal to the theoretical distribution. As our distance focus on the tails of the distribution, we can say this is a “tail-goodness-of-fit” test.

<sup>3</sup>We will call them CD distance and FOF distance, respectively.

<sup>4</sup>This is important when short positions on the asset are frequent.

**5. An Empirical Application.** In this section, data from three major exchange rates are used to exemplify the tail-goodness-of-fit test proposed on last section. We will use data from three major exchange rates, and test 5 distributions: Normal, Scaled-t, GH, NIG and Hyperbolic.

**5.1. Data Description.** The three exchange rates used are Japanese Yen (JPY) per U.S. Dollar, Swiss Franc (CHF) per U.S. Dollar and U.S. Dollar per Great Britain Pound (GBP). It was used daily close bid prices from January 1<sup>st</sup> 1987 to August 29<sup>th</sup> 2002. The returns used were logarithmic, according to the following formula:

$$R_t = \ln \left[ \frac{CloseBid_t}{CloseBid_{t-1}} \right]$$

Where  $R_t$  is the return from day  $t$ ,  $CloseBid_t$  is close bid price from day  $t$ , and  $Ln$  is neperian logarithmic.

In table I, we have the main information about the return sample used:

Table I: Sample Return Statistics

	<i>Japanese Yen</i>	<i>Great Britain Pound</i>	<i>Swiss Franc</i>
Mean	-0.0070%	0.0011%	-0.0019%
Standard Deviation	0.727%	0.616%	0.739%
Kurtosis	5.16530	3.21694	1.81152
Asymmetry	(0.49884)	(0.19496)	(0.09836)
Range	10.99%	7.69%	7.39%
Minimum	-6.95%	-4.39%	-3.55%
Maximum	4.04%	3.30%	3.84%
Number of Observations	4,083	4,083	4,083

We can see on graphs I to III that the tails of the empirical histogram are heavier than the Normal distribution estimated, for all three currencies. The density around the mean is also higher in the empirical distribution.

**5.2. Estimation of the parameters.** Before performing the hypothesis test, we need first to estimate the parameters of the theoretical distributions. The parameters of Scaled-t, GH, NIG and Hyperbolic were estimated using maximum log-likelihood. Estimated parameters are on table II. The density functions of these distributions are on Appendix A.

**5.3. Goodness-of-fit Tests of Major Exchange Rates.** After estimating the parameters, the next step to perform the test is to calculate the critical values and the p-values to each set of parameters, considering the sample size of 4,083.

The critical values were obtained after 10.000 Monte Carlo (MC) Simulations runs for our sample size. For each MC run, we do the following two steps:

Generate a sample of the size proposed, using the distribution and parameters proposed;

Calculate the Bi-caudal CvaR (BCV) distance between the sample generated on step 1, with the theoretical distribution;

At the end of the 10.000 runs, we will have 10.000 BCV distances that will be ordered. So the critical value for the significance level  $S$  will be the percentile  $(1 - S)$  of the ordered BCV distance sequence.



Graph I: Histogram - Japanese Yen x U.S. Dollar Returns

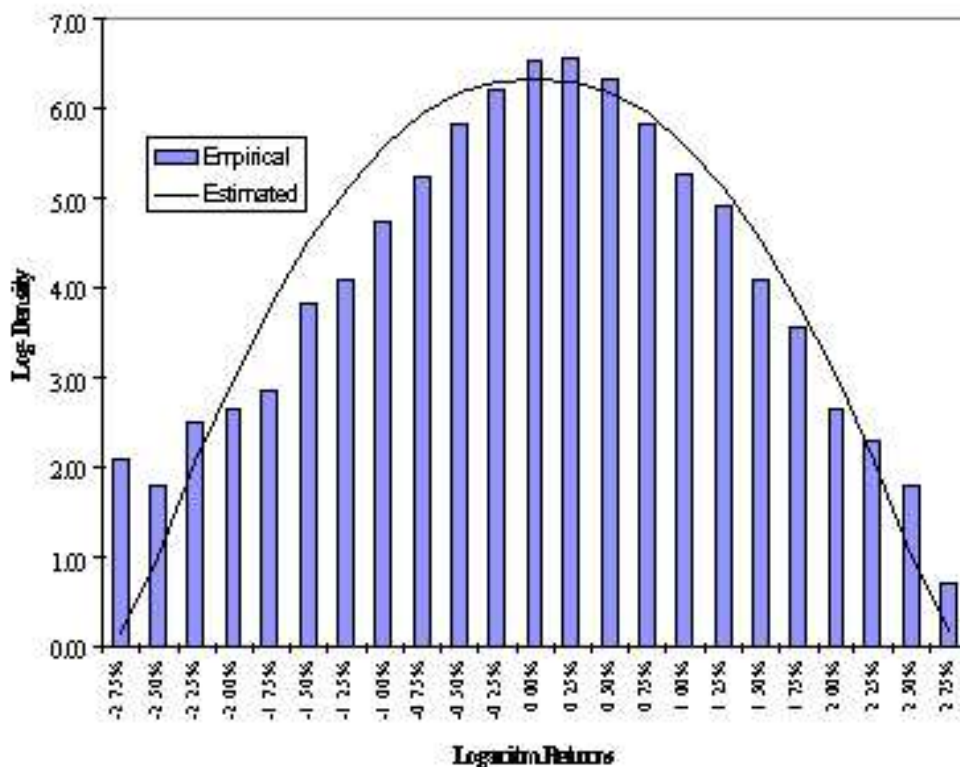


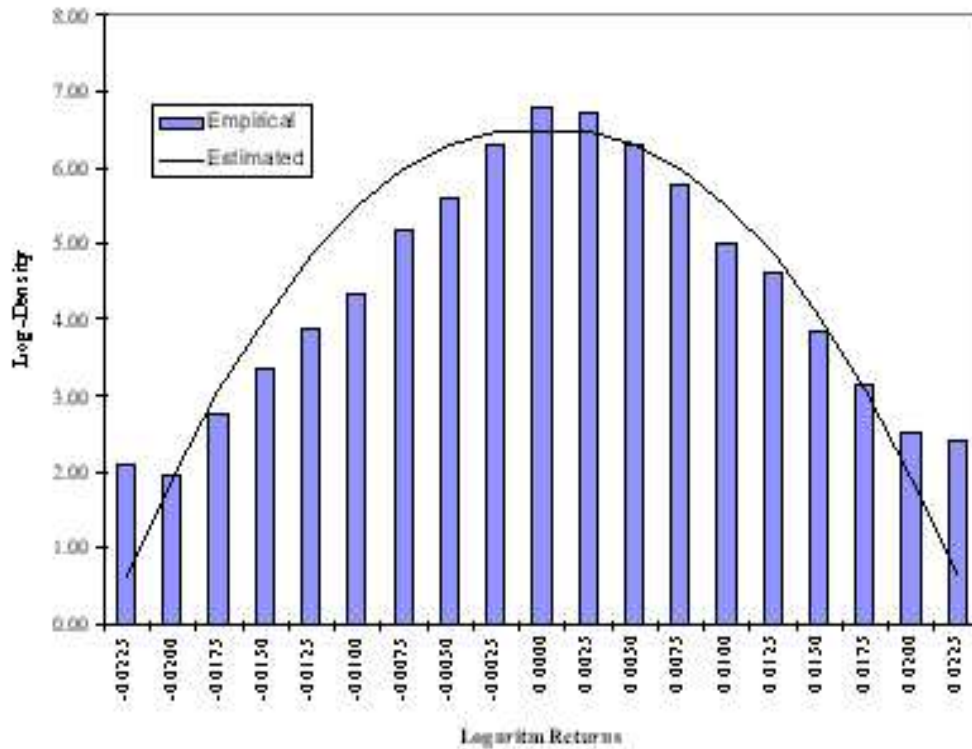
Table II: Estimated Parameters

		Alfa	Beta	Delta	Mi	Lambda	LogL.
GH	GBP	1.82E+02	-4.49E+00	4.30E-03	1.81E-04	4.90E-02	15,235
	CHF	2.46E+02	-6.12E+00	2.81E-03	3.14E-04	1.51E+00	14,364
	JPY	7.36E+01	-1.58E+01	9.69E-03	7.47E-04	-1.59E+00	14,571
NIG	GBP	1.46E+02	-4.53E+00	5.56E-03	1.83E-04	-5.00E-01	15,234
	CHF	1.63E+02	-7.43E+00	8.91E-03	3.88E-04	-5.00E-01	14,363
	JPY	1.39E+02	-1.65E+01	7.07E-03	7.74E-04	-5.00E-01	14,569
Hyperb	GBP	2.46E+02	-4.36E+00	1.82E-03	1.73E-04	1.00E+00	15,233
	CHF	2.25E+02	-6.54E+00	4.74E-03	3.38E-04	1.00E+00	14,364
	JPY	2.17E+02	9.00E-03	3.02E-03	1.51E-04	1.00E+00	14,555
Scaled-t		Mi	Sigma	DoF	LogL.		
	GBP	6.43E-05	6.40E-03	3.81	15,226		
	CHF	5.36E-05	7.46E-03	5.46	14,359		
	JPY	1.33E-04	7.35E-03	4.21	14,564		

Then, to obtain the p-value, we have just the find the significance level corresponding to the BCV distance calculated between the empirical and theoretical distribution, using the parameters from Table II.

Then we can compare the p-value to a certain significance level (we choose 1%), to assess the null hypothesis that the empirical distribution is equal to the theoretical distribution.

Graph II: Histogram - Great Britain Pound x U.S. Dollar Returns



These results are shown on table III:

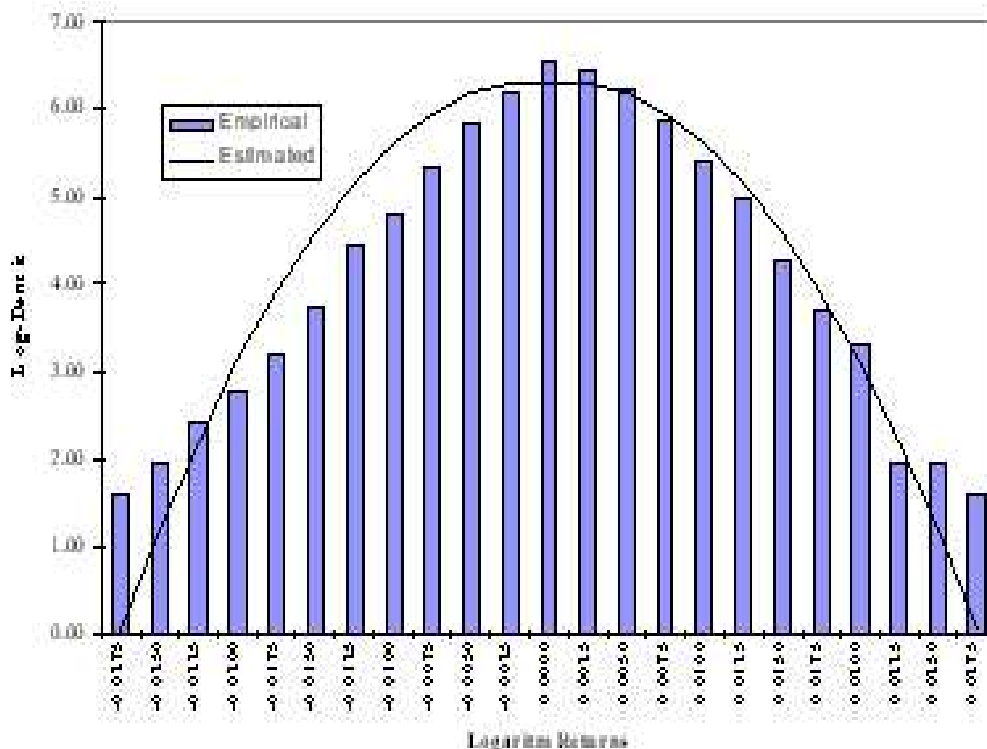
Table III: BCV Hypothesis Tests Results

	JPY		GBP		CHF	
	BCV Dist.	p-value	BCV Dist.	p-value	BCV Dist.	p-value
Normal	3.08E-03	0.000%	3.16E-03	0.000%	2.73E-03	0.000%
Scaled-t	2.53E-03	0.970%	5.82E-04	69.97%	1.10E-03	23.07%
NIG	5.37E-04	68.14%	<b>3.32E-04</b>	84.21%	<b>2.97E-04</b>	87.74%
Hyperbolic	2.90E-03	0.000%	5.72E-04	50.30%	4.77E-04	69.07%
GH	<b>3.16E-04</b>	89.00%	4.46E-04	71.69%	5.32E-04	62.17%

As expected, results from table III had rejected normality of all three exchange rates tested, corroborating previous empirical studies. For the CHF and GBP, all the others distribution have been not rejected on the BCV hypothesis test, but for the JPY, only the NIG and GH were not rejected with 1% confidence level, although the scaled-t is very near the confidence level.

The best distribution according to the minimum BCV distance criteria is the NIG for GBP and CHF, and the GH for the JPY. Using the Maximum LogLikelihood (MLL) criteria, the GH is the best one for all currencies, but it is expected, since it is a general case of the others (see Appendix A), and the estimation procedure tried to maximize the LogLikelihood function. Since the BCV distance has focus on the tails of the distribution and the estimation used here (MLL) considers all parts of the

Graph III: Histogram - Swiss Franc x U.S. Dollar Returns



distribution, even with the GH being a general case, it will not have necessarily the lowest BCV distance.

To provide a comparison to classical tests, the Kolmogorov-Smirnov and Kuiper tests were also performed on the same data as can be seen on tables IV and V:

Table IV: Kolmogorov-Smirnov Hypothesis Test Results

	JPY		GBP		CHF	
	Distance	p-value	Distance	p-value	Distance	p-value
Normal	5.60E-02	0.000%	5.60E-02	0.000%	4.40E-02	0.000%
Scaled-t	1.28E-02	51.44%	1.25E-02	54.76%	1.19E-02	60.96%
NIG	<b>6.92E-03</b>	98.95%	9.13E-03	88.38%	9.81E-03	82.48%
Hyperbolic	1.15E-02	64.83%	<b>9.11E-03</b>	88.60%	9.06E-03	88.97%
GH	8.58E-03	92.37%	9.13E-03	88.45%	<b>8.85E-03</b>	90.49%

The classical tests also rejected Normality as was expected, with the all other distributions being not rejected. Note that for the CHF and GBP, test results are the same of the BCV, but for the JPY, the Scaled-t and the Hyperbolic that were rejected on the BCV test, now are being not rejected. So we can say that, for overall applications, we can use Scaled-t and Hyperbolic distributions for the JPY, but for Risk Management applications, they are not adequate.

Results about the best distribution for these criteria are mixed: for the JPY currency, the NIG was the best distribution on both Kolmogorov and Kuiper distances. For the GBP and CHF, Hyperbolic

Table V: Kuiper Hypothesis Test Results

	JPY		GBP		CHF	
	Distance	p-value	Distance	p-value	Distance	p-value
Normal	1.04E-01	0.000%	1.11E-01	0.000%	7.80E-02	0.000%
Scaled-t	2.41E-02	14.63%	2.12E-02	31.96%	2.31E-02	19.59%
NIG	<b>1.32E-02</b>	95.75%	1.65E-02	74.45%	1.76E-02	64.09%
Hyperbolic	2.05E-02	37.15%	1.76E-02	64.83%	<b>1.59E-02</b>	79.73%
GH	1.70E-02	70.31%	<b>1.58E-02</b>	81.01%	1.62E-02	77.36%

and GH were the best distribution depending on the criteria (Kolmogorov or Kuiper).

**6. Conclusion and Suggestions.** In this paper we proposed a way to test the goodness-of-fit of theoretical distributions to empirical data with focus on the CVaR risk measure. The JPY, GBP and CHF exchange rates were used as examples of a practical application of the test proposed. These tests were done considering Normal, Scaled-t, Hyperbolic, NIG and GH distributions, and three distances criteria, Kolmogorov, Kuiper and the BCV distance.

For JPY currency, Normal, Hyperbolic and scaled-t distributions were rejected by the BCV test. For the CHF and GBP, only Normality was rejected. When using classical tests (Kolmogorov-Smirnov and Kuiper) test results are the same for CHF and GBP. But for the JPY, the Scaled-t and the Hyperbolic are rejected on the BCV test, and not rejected for the other two tests. We may conclude that, for overall finance applications, we can use Scaled-t and Hyperbolic distributions for the JPY, but for Risk Management applications, they are not adequate.

The test proposed on this article can be easily applied to other kinds of distributions and assets, including portfolios, since it needs only the series of returns and the expression for the distribution's density. So, as suggestion for further researches, another kinds of distributions may be also tested, together with other classes of assets.

It is worth to mention that the test proposed here assesses the unconditional distribution of returns, but most of the risk management approaches use conditional distributions approaches. Anyway, as suggested by Berkowitz (2002, figure 1), one of the steps to check the validity of a risk model is to test the distribution used, and this would be done by the CVaR test proposed on this paper.

Also the CVaR-focuses distance proposed can be used to estimate the parameters of distributions, through minimization of this distance. For example, Prause (1999) uses an estimation method that minimizes the Anderson-Darling distance. Then, with an estimation focused on the tails of the distribution, the CVaR measure is expected to be more reliable. After the distance minimization estimation, a conditional volatility model may be used to re-scale the distribution to get the volatility forecasted by the model. Backtests would then assess the validity of this approach.

## Appendix A

**A.1 Stable Paretian distributions** Characteristic function of the Stable Paretian distribution is the following:

$$\phi_x(t) = \exp \{-\gamma|t|^\alpha (1 - i\beta \text{sign}(t) \text{tg}(\pi\alpha/2)) + it\delta\} \text{ for } \alpha \neq 1$$

and

$$\phi_x(t) = \exp \{-\gamma|t| (1 + 2i\beta \text{sign}(t) \ln(|t|)) / \pi + it\delta\} \text{ for } \alpha = 1$$

Where  $\delta$  is the location parameter,  $\gamma$  is the scale/dispersion parameter,  $\beta$  is the asymmetry parameter and  $\alpha$  is the stability index.

**A.2 Scaled Student t distribution** The density function of the scaled Student t distribution is the following:

$$f(x; \mu, \sigma, \nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\nu/2)\sqrt{\pi(\nu-2)\sigma^2}} \left[ 1 + \frac{(x-\mu)^2}{(\nu-2)\sigma^2} \right]^{-(\nu+1)/2}$$

Where  $\nu$  is the degrees of freedom parameter,  $\mu$  is the location parameter and  $\sigma$  the dispersion parameter.

When  $\nu \rightarrow \infty$  the Student t converges to the Normal Distribution.

**A.3 Mixture of Normals** Density function of the returns is a weighted average of several Normal densities:

$$f(x; \mu_1, \dots, \mu_n, \sigma_1, \dots, \sigma_n, \lambda_1, \dots, \lambda_{n-1}) = \sum_{i=1}^n \lambda_i N(\mu_i, \sigma_i)$$

Where  $\mu_i$  and  $\sigma_i$  are respectively the mean and standard deviation of each Normal distribution, and  $\lambda$  is the weight of each Normal. As the sum of the weights must be one, the last  $\lambda$  is completely defined by the others.

**A.4 Generalized Hyperbolic Distribution** The density probability function of one-dimensional GH distribution is defined by the following equation:

$$GH(x; \lambda, \alpha, \beta, \delta, \mu) = a(\lambda, \alpha, \beta, \delta) (\delta^2 + (x - \mu)^2)^{(\lambda-1/2)/2} \times K_{\lambda-1/2} \left( \alpha \sqrt{\delta^2 + (x - \mu)^2} \right) e^{\beta(x-\mu)}$$

where  $K_x$  is the modified Bessel function of third kind and

$$a(\lambda, \alpha, \beta, \delta) = \frac{(\alpha^2 - \beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-0.5}\delta^\lambda K_\lambda \left( \delta \sqrt{\alpha^2 - \beta^2} \right)}$$

The parameters are real numbers with the following restrictions (see Prause[1999]):

$$\begin{array}{lll} \delta \geq 0, & |\beta| < \alpha & \text{if } \lambda > 0 \\ \delta > 0, & |\beta| < \alpha & \text{if } \lambda = 0 \\ \delta > 0, & |\beta| \leq \alpha & \text{if } \lambda < 0 \end{array}$$

The parameter  $\delta$  is a scale factor, compared to the  $\sigma$  of a Normal distribution, and  $\mu$  is a location parameter. Parameters  $\alpha$  and  $\beta$  determine the distribution shape and  $\lambda$  defines the subclasses of GH and

is directly related to tail fatness (Barndorff-Nielsen and Blæsild [1981]). The function  $a(\cdot)$  is introduced to guarantee that the cumulative distribution has values between zero and one.

The GH has several subclasses, among them the Hyperbolic and Normal Inverse Gaussian(NIG). Setting  $\lambda = -1/2$ , we get the NIG, and with  $\lambda = 1$ , we get the Hyperbolic distribution. The Gaussian is a limiting distribution of GH, when  $\delta \rightarrow \infty$  and  $\delta/\alpha \rightarrow \sigma^2$ .

## References

- [1] Anderson, T. W. and D. A. Darling (1952) Asymptotic Theory of Certain Goodness of Fit Criteria, *Annals of Mathematical Statistics*, Vol. 23, Issue 2, p. 193-212.
- [2] Artzner, P., F. Delbaen, J. Eber and D. Heath (1999) Coherent Measures of Risk, *Mathematical Finance*, Vol. 9, p. 203-228.
- [3] Barndorff-Nielsen, O. and P. Blæsild (1981) Hyperbolic Distributions and Ramifications: Contributions to Theory and Application, *Statistical Distributions in Scientific Work*, Vol. 4, 19-44.
- [4] Basak, S. and A. Shapiro (2001) Value-at-Risk-Based Risk Management: Optimal Policies and Asset Prices, *The Review of Financial Studies*, Vol. 14, N° 2, p. 371-405.
- [5] Berkowitz, J. (2002) Testing the Distributional Assumptions Embedded in Risk Models, *Risk*, May 2002, available at <http://www.uh.edu/~jberkowi/disttest.pdf>
- [6] Blattberg, R. and N. Gonedes (1974) "A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices" *Journal of Business*, 47, 244-280.
- [7] Brauer, C. (2000) Value at Risk Using Hyperbolic Distributions, *Journal of Economics and Business*, vol. 52, pp. 455-467.
- [8] Courtault J., Y. Kabanov, B. Bru, P. Crépel, I. Lebon and A. Le Marchand (2000) Louis Bachelier On The Centenary Of Théorie de la Spéculation, *Mathematical Finance*, Vol.10, 341-353.
- [9] Crnkovic, C. and Drachman, J. (1996) Quality Control, *Risk*, vol. 9; n° 9, September 2001, p. 138-143.
- [10] Eberlein, E. e Keller, U. (1995) Hyperbolic Distributions in Finance, *Bernoulli* 1, 281-299.
- [11] Embrechts P, Klueppelberg, C, Mikosch, T (1997) Modelling Extremal Events for Insurance and Finance. Springer-Verlag. Berlin.
- [12] Fama, E. F. (1965) The behavior of stock market prices, *Journal of Business*, 38, p. 34-105.
- [13] Farias, A., Ornelas, J. and Fajardo, J. (2003) Analyzing the Use of Generalized Hyperbolic Distributions to Value at Risk Calculations, Annals of the III Brazilian Finance Meeting.
- [14] Hull, J. and A. White (1998) Value At Risk When Daily Changes in Market Variables are Not Normally Distributed, *Journal of Derivatives*, Vol. 5, N° 3, pp 9-19.
- [15] Jorion, P. (2001) Value at Risk: The New Benchmark for Controlling Market Risk, McGraw-Hill, New York.
- [16] Kon, S. J. (1984) Models of stock returns - A comparison. *The Journal of Finance*, 39,1, 147-165.
- [17] Kuiper, N. H. (1962) Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Ser. A, vol. 63, pp 38-47.
- [18] Mandelbrot, B. B. (1963). The Variation of Certain Speculative Prices. *The Journal of Business*, Vol. 36, p. 394-419.

- [19] Massey, F. J. (1951) The Kolmogorov-Smirnov Test for goodness of fit, *Journal of the American Statistical Association* 46, 68-78.
- [20] Praetz, P. (1972) "The Distribution of Share Price Changes." *Journal of Business*, 45, 49-55.
- [21] Prause, K. (1999) The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures, PhD Dissertation, University of Freiburg.
- [22] Osborne, M. F. M. (1959) Brownian motion in the stock market, *Operations Research*, Vol. 7, p. 145-173.
- [23] RISKMETRICS<sup>TM</sup> (1995, 1996) *Technical Document* – J.P.Morgan / Reuters.
- [24] Rydberg, T. H. (1997) Why Financial Data are Interesting to Statisticians, *Working Paper 5*, Centre for Analytical Finance of the university of Aarhus, 1997.
- [25] Samuelson, P. (1965) Rational theory of warrant pricing. *Industrial Management Review*, Vol. 6,p. 13-32.