Model Risk in Option Pricing

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Abstract: Several generalisations of the Black–Scholes (BS) Model have been made in the literature to overcome the well–known empirical inadequacies of the BS–Model. In this work I perform an empirical comparison of stochastic volatility models established by Duffie et al. (2000) with jumps in the volatility and four deductive special cases. In addition I include the model of Schoebel/Zhu (1999) with volatility driven by an Ornstein–Uhlenbeck process instead of a Cox–Ingersoll–Ross process. As Zhu (2000) suggested the model can be easily combined with a jump component in the underlying. I examine the resulting model empirically and stress its good properties. This comparison embeds out–of–sample pricing performance as an important element in a model performance study based on model risk. The main result in terms of fit performance is that the most complex models are not always the best ones. It is important to quantify model risk like e.g. Cont (2004) and to examine the sensitivity of exotic options in terms of moneyness, maturity and market condition. To achieve this comparison the model risk measure of Cont (2004) is extended and applied to various exotic options.

Keywords: model risk; option pricing; stochastic volatility; model comparison; calibration

1 Introduction

A central object of the capital market research is the development and empirical check-up of derivatives pricing models. Black/Scholes (1973) and Merton (1985) achieved ground breaking findings in the seventies of the twentieth century with their celebrated option pricing model (below: (constant volatility) CV model). In empirical examinations, however, it turned out that the "stylised facts"[a] observed at the market could not be represented well. In order to correspond better to the market facts, the necessity originated to relax some of the very restrictive assumptions of the CV model. On that occasion the starting point was to model a not–constant volatility and jumps in the underlying instrument. Numerous new models were created by the combination of both approaches.
The large number of alternative pricing models makes it difficult for option writers and buyers to choose an adequate model. If an investor selects the "wrong" model he could sustain a loss that arises exclusively from his model choice. Since this model risk can be found in each model, it should be conscious for the investor.

[a]among other things: volatility smile, heavy tails

The market determines the prices of standard options traded on a stock exchange. So in this case the model risk is more or less irrelevant. However, it is especially important for the pricing of ”over–the–counter” (below: OTC) exotic options since these don’t have any market prices.
In order to examine the model risk in this paper, ten different option pricing models are calibrated at observed market prices by the minimisation of the quadratic distance from market price to model price. The data record for the empirical study contains market prices on the DAX, the German stock index, from the years 2002 to 2005.
After the calibration, exotic options are calculated using the optimal parameters. The resultant prices are strongly different despite calibration. If the investor decides in favour of one of these prices, he takes automatically model risk. Since it is known that model risk exists, it is important to know which model has good qualities in several dimensions like in–sample fitting and out–of–sample performance. In order to minimise the model risk the investor should rely on the best model in the mentioned categories. In this paper, a multidimensional comparison of option pricing models is performed.
It is obvious that each exotic option has a different model risk and that model risk could be larger in bullish or bearish markets. So one objective of this paper is to study how sensitive a certain exotic option is to model risk.
The contributions of my paper to the current literature are: first, a very general specification of stochastic volatility models. Second, I propose three new stochastic volatility models using the OU–process that are very successful in the empirically tests and have lower pricing errors than their competitors using CIR–processes. Thirdly, comprehensive in– and out–of–sample performance tests are provided. Finally, the model risk, measured by a new model risk measure, of different exotic options is studied.
The remainder of the paper is structured as follows. In Section 2 the option pricing models used in this study are introduced and the new expansions of the models are developed. Section 3 evaluates the in–sample and out–of–sample fits to the market data, while Section 4 reports the pricing of the exotic options. In Section 5 the model risk is studied on the basis of a new model risk measure. Section 6 summarises and concludes.

2 Stochastic volatility models

2.1 General model specification

The option pricing models used in this paper belongs to an affine jump diffusion family. The following stochastic differential equation (1) describes a very general class of this family. All later relevant models are special cases of this stochastic differential equation under the risk–neutral probability measure $\mathbb{Q}$:

$$
\begin{aligned}
dS_t &= (r - \lambda_S \bar{\mu}) S_t dt + V_t^p S_t dW_t^{(S)} + J_S S_t dN_t^{(S)} \\
dV_t &= \kappa(\theta - V_t) dt + \beta V_t^{1-p} dW_t^{(V)} + J_V dN_t^{(V)} \\
dW^{(S)} dW^{(V)} &= \rho dt.
\end{aligned}
\tag{1}
$$

Let S_t denote the price process for the non–dividend paying underlying. Its instantaneous volatility V_t is given by a mean–reverting process with κ as the rate of mean reversion, θ as the long term mean and β as the volatility of the process.
Mean–reverting means that if the current level is above the long term mean θ, the process will tent towards θ. On the other hand if the process is below the level θ, the process will drift up to θ.
The two processes, $W^{(S)}$ and $W^{(V)}$ are standard Brownian motions with correlation coefficient ρ. In equation (1) the underlying price process is modelled as a diffusion with an added Poisson jump component $N_t^{(S)}$ with intensity λ_S and $(1 + J_S) \sim$ log–normal(μ_S, δ_S).
Beside the jumps in the underlying also jumps in the volatility are taken into account in (1). The process $N_t^{(V)}$ is also a Poisson jump process with intensity λ_V. The size of the jumps is exponentially distributed with parameter μ_V: $J_V \sim \text{Exp}(\mu_V)$. Despite jumps in volatility the mean–reverting quality remains valid: if the process jumps away from the long term mean, the path will decline again to the mean.
Jumps in volatility are a useful property since the volatility escalate after a downwards jump in the underlying. Without the possibility to jump the volatility path lasts longer to reach the new level. Furthermore it is observed that without jumps the variance parameter β takes unrealistically high values.
It is assumed that a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F})_{0\le t\le\infty}$ exits. Furthermore it is necessary to know that an equivalent risk–neutral measure $\mathbb{Q}$ exits to the probability measure $\mathbb{P}$. More technical details to the existence of such a measure can be found in Hull (2000). To ensure that the equation (1) is in fact under the risk–neutral measure for each model, the parameter $\bar{\mu}$ must be chosen correctly. With $p = \frac{1}{2}$ in equation (1) the framework of Duffie et al. (2000) is chosen. With $p = 1$ the framework of Zhu (2000) is selected. The respective properties of both frameworks are described in the following sections.

2.2 Framework of Duffie, Pan and Singleton (2000)

Duffie et al. (2000) introduced two affine "double jump" models. Both models have in common that the variance is modelled by a square–root diffusion process:

$$dV_t = \kappa(\theta - V_t)dt + \beta\sqrt{V_t}dW_t^{(V)} + J_V dN_t^{(V)}. \tag{2}$$

The similar process without jumps

$$dV_t = \kappa(\theta - V_t)dt + \beta\sqrt{V_t}dW_t \tag{3}$$

was proposed for stochastic interest rates by Cox et al. (1985) (below: CIR). Feller (1951) showed that the variance V_t is always non–negative and if

$$2\kappa\theta > \beta^2 \tag{4}$$

the process can never reach zero. Equation (4) is used as a constraint in the following study to get reasonable parameters especially for the discretisation schemes[b].
The first affine double jump model has independent jumps (below: IJ) in the underlying and in the volatility. In the following this model is called "SVJ-IJ-CIR".

[b]see e.g. Lord et al. (2006)

Table 1 Option pricing models with CIR volatility process: the number of the parameters, the characteristics and the authors who published the model first

	# parameters	characteristics	authors
SVJ-IJ-CIR	10	$N^{(S)}$ (λ_S) and $N^{(V)}$ (λ_V) independent μ_V, μ_S, δ_S, CIR (V_0, κ, θ, β), ρ	Duffie et al. (2000)
SVJ-CJ-CIR	10	$N^{(V)} = N^{(S)}$, corr. jumps with ρ_V λ_V, μ_V, μ_S, δ_S, CIR, ρ	Duffie et al. (2000)
SVJ-CIR	8	$N^{(V)} = 0$, $\lambda_V = 0$ λ_S, μ_S, δ_S, CIR ρ	Bates (1996)
SV-CIR	5	$N^{(V)} = N^{(S)} = 0$ CIR, ρ	Heston (1993)
CVJ	4	constant volatility $V_t \equiv V$ λ_S, μ_S, δ_S	Merton (1976)
CV	1	constant volatility σ	Black/Scholes (1973)

So there are two Poisson processes $N_t^{(S)}$ and $N_t^{(V)}$ with different intensities λ_S and λ_V. Altogether the SV-IJ-CIR model has ten parameters: V_0, κ, θ, β, ρ, λ_S, μ_S, δ_S, λ_V and μ_V.
The second model has correlated jumps (below: CJ): a jump in the volatility is always followed by a jump in the underlying. The underlying's jump size is correlated to the jump size in the volatility with the correlation coefficient ρ_V that is normally negative. In analogy this model is called "SVJ-CJ-CIR" and has also ten parameters: V_0, κ, θ, β, ρ, λ_S, μ_S, δ_S, ρ_V and μ_V.
Four popular option pricing models can be deduced from Duffie et al's framework. Let $\lambda_V = 0$ then the variance process in equation (2) reduces to the CIR process in (3). This model invented by Bates (1996) has still jumps in the underlying and therefore eight parameters. In this paper the Bates' model is labeled as "SVJ-CIR". With $\lambda_S = \lambda_V = 0$ all jump components are deleted. The resultant model was introduced by Heston (1993). It has five parameters (V_0, κ, θ, β, ρ) and is called "SV-CIR" model.
Merton (1976) introduced a option pricing model with constant volatility $V_t \equiv \sigma$ and jumps in the underlying. The so called "CVJ" model has four parameters: λ_S, μ_S, δ_S, σ. The last model with only one parameter σ is the well–known Black–Scholes model "CV". A summary of the six models can be found in Table 1.

2.3 Framework of Schoebel and Zhu (1999)

Schoebel/Zhu (1999) model stochastic volatility as an Ornstein–Uhlenbeck process (Ornstein/Uhlenbeck (1930)) (below: OU)

$$dV_t = \kappa(\theta - V_t)dt + \beta dW_t^{(V)}. \tag{5}$$

Since $p = 1$ in equation (1) the volatility instead of the variance is modelled by equation (5). To describe the stochastic volatility using an OU process occurs less frequently in the literature than using a CIR process. This has a main reason: The OU process may become negative. On the first sight that is a problem, in particular

for interest rates. As Vasicek (1977) proposed an interest rates model with an OU process, the negative values were the big drawback. But in order to model volatility it is important to see volatility as an additional parameter with its own co–domain. To interpret volatility as standard deviation is only correct in the Black–Scholes' world. Anyway the variance as squared volatility is positive in any case.
The advantages of the OU process are clearly obvious. First of all the associated variance process is a mean–reverting process, too[c]. For the CIR process the volatility process is given by

$$dV_t = -\gamma V_t dt + \beta dW_t. \tag{6}$$

Applying the Itô formula to equation (6) yields the familiar variance $y(t) = V_t^2$ with $\gamma = \frac{\kappa_h}{2}$, $\beta = \frac{\beta_h}{2}$, $\theta_h = \frac{\beta^2}{\kappa_h}$ and

$$dy(t) = \kappa_h(\theta_h - y(t))dt + \beta_h \sqrt{y(t)} dW_t. \tag{7}$$

Equation (6) is an OU process with $\theta = 0$. So it is not a mean–reverting process. Secondly, under the risk–neutral measure the OU process is normal distributed and explicit solvable. Finally, with regard to discretise and simulate the process it has an exact time–discrete scheme. All these points are not the case for the CIR process.
In order to use these good properties the model of Schoebel and Zhu is added to the comparison. Specified as "SV-OU" with the five parameters V_0, κ, θ, β and ρ it is a direct competitor to the SV-CIR model.
In addition to the original SV-OU model I have combined the OU volatility with jumps in the underlying process explicitly. Schoebel/Zhu (1999) mentioned the model already in this paper as a part of a modular system. However, an explicit examination and empirical study of the model were still missing. The model is called "SVJ-OU" and has eight parameter: the five volatility parameters mentioned above and the three jump–parameters λ_S, μ_S and δ_S. Consequently it competes with the SVJ-CIR model.
A complex model with more parameters seems preferable, because it should describe the market better. But in general complex models are more sophisticated to implement and more difficult to estimate. For these reasons it could be an advantage to use a model with few parameters and good performance. A method to reduce one parameter in the SV-OU models offers equation (7), because the parameters are over–determined. The long term mean θ could be set to zero in the SV-OU model, but in the SV-CIR it remains non–zero. Taking advantage of this relation I have specified a full mean–reverting model, "SV4" with only four parameters (V_0, κ, β, ρ) left.
Finally I have combined this approach with jumps in the underlying. This ends up in a new jump–diffusion model "SVJ7" with seven parameters: V_0, κ, β, ρ, λ_S, μ_S and δ_S.
All four models are summarised in Table 2. Altogether I have come up with four models that were never or only rarely examined in the literature so far. In the next section I provide such a comparison.

[c]see Schoebel/Zhu (1999)

Table 2 Option pricing models with OU volatility process: the number of the parameters, the characteristics and the authors who published the model first

	# parameters	characteristics	authors
SV-OU	5	OU (V_0, κ, θ, β), ρ	Schoebel/Zhu (1999)
SVJ-OU	8	SV-OU with jumps in the underlying λ_S, μ_S, δ_S, OU, ρ	Zhu (2000) (modular)
SV4	4	SV-OU with $\theta = 0$ OU^{res} (V_0, κ, β), ρ	
SVJ7	7	SV4 with jumps in the underlying λ_S, μ_S, δ_S, OU^{res}, ρ	

3 In–sample and out–of–sample performance

3.1 Pricing and calibration

The purpose of this section is to give a composition of the in–sample and out–of–sample performance of option pricing models described in Section 2. In this study the calibration method to obtain the model parameters is to imply the parameter from observed option prices. Option prices are suitable estimators for the models, because option prices reflect the beliefs in future trends.

On each trading day the model is recalibrated using the cross–sectional option price data. A daily recalibration is consistent with parameter estimation in practice. So the parameters are non–constant over time and time series for any parameter are the result. Calibration means in detail to solve the following minimisation problem

$$\Theta^* = \arg\min_{\Theta} \sum_{j=1}^{k} \sum_{i=1}^{s(k)} \left(V^{\text{market}}(K_j, T_i(k)) - V^{\text{model}}(K_j, T_i(k)) \right)^2 . \tag{8}$$

The model depends on the set of parameters $\Theta = (p_1, p_2, \dots, p_n)$ that were introduced in section 2. The observed market price of plain vanilla options is given by V^{market} and the corresponding model price is written by V^{model}. In order to find an optimal Θ each trading day all observed option prices with k strike levels K_j and $s(k)$ maturity times $T_i(k)$ are used that day.

In equation (8) the objective function is to minimise the sum of squared errors. It should be noticed that different objective functions result in a dissimilar parameter Θ and in different in–sample fit performance[d]. I have chosen the above function (8) because this one is widely spread in the literature.

The optimisation problem (8) is hard to solve, because the objective function is non-linear and there are many local minima that should not be chosen. To overcome these problems I have done a global simulated annealing minimisation[e] first followed by a local quasi Newton method[f]. Several tests with different starting values maintain the results in this section. A showcase how the calibration works is given in Figure 1.

[Figure 1 here]

[d] see e.g. Detlefsen/Haerdle (2006)
[e] see e.g. Cerny (1985)
[f] see e.g. Broyden (1969)

Beside an accurate optimisation algorithm a fast method to calculate the risk–neutral price V_t for a European plain vanilla call with terminal payoff $V_T = \max(S_T - K, 0)$ as

$$V_t = E^{\mathbb{Q}}\left(e^{-r(T-t)}V_T\right) \tag{9}$$

is needed. To evaluate the expectation in equation (9) Carr/Madan (1998) used the characteristic function

$$\phi_T(u) = E^{\mathbb{Q}}\left(e^{i\cdot u\cdot \log S_T}\right). \tag{10}$$

and Fast Fourier Transformation. So closed form solutions for option prices are obtained via Fourier inversion. In this paper the characteristic functions are known in closed form for each relevant model.

3.2 Data description

The German Stock Index "DAX30" is chosen as underlying in my empirical work. The DAX30 is composed of thirty most important blue chips in Germany. To consider index options has three main advantages. First, an index acts better as an indicator for the economy than an individual stock option does. Secondly, index options are traded liquidly in large numbers. Finally, DAX30 index options are European–style contracts and don't pay any dividends.
The dataset contains daily closing prices of the DAX30 and the put and call closing prices of the corresponding plain vanilla options traded on the EUREX from January 2002 to September 2005[g]. Altogether the data period covers 956 trading days. Due to the put–call parity only the call prices are used to calibrate the models. Pricing errors based on the put prices are similar in their explanatory power and so skipped in this paper.
From the original dataset some options with infrequently trades and liquidity–related bias are excluded. These are options with a price below one Euro or with less than ten or more than 510 trading days to maturity. The moneyness m of an option is here defined as $m = K/S_0$ with current underlying S_0 and strike K. Options to deep in–the–money with moneyness $m < 0.75$ or to deep out–of–money with $m > 1.35$ are erased as well. Finally, only options are selected that remain within the arbitrage bounds and fulfil the put–call–parity

$$V_t^{\text{Put}} = V_t^{\text{Call}} + K\exp(-r(T-t)) - S_t. \tag{11}$$

After the screening the dataset contains 158,755 options with a mean of 166 prices per trading day.
As a proxy for the risk free interest rate the six–month EURIBOR is selected. The time series for the DAX30 and the EURIBOR are plotted in Figure 2.

[Figure 2 here]

[g]Data origin: Karlsruher Kapitalmarktdatenbank, University of Karlsruhe, Germany

Table 3 The mean value of β in different trend periods

Trading days	overall period 1 to 956	bearish market 100 to 200	bullish market 300 to 400	trendless 600 to 700
SV-CIR	0.47	0.72	0.58	0.43
SVJ-CIR	0.35	0.65	0.44	0.29
SVJ-IJ-CIR	0.31	0.61	0.37	0.26
SVJ-CJ-CIR	0.32	0.58	0.36	0.26

3.3 Estimated parameters

As described in Section 3.1 the model parameters for each model are estimated using equation (8). The ten models have altogether 63 different parameters. So the calibration yields to 63 time series. It is impossible and not needed anyway to show each time series in this paper. However, some special parameters and facts are reported in the following.
In each model the correlation ρ of the underlying and the volatility is always negative. This is consistent with the results in the literature. A down movement in the underlying goes along with an up movement in the volatility and the other way round.
Eraker et al. (2003) argued that the estimated volatility β of the volatility is too high to be in the line with time series estimates of the volatility. But with the addition of jumps in the volatility β can be reduced to a matching level. In Table 3 the mean values of the parameter β in different models and market periods is presented. The β in the SV-CIR model is actually the highest value. Adding jumps reduces β by a third in the SVJ-IJ-CIR and SVJ-CJ-CIR model. Thus my findings are in line with Eraker's results. But the same effect can be generated by adding jumps only to the underlying and not to the volatility process. The β in the SVJ-CIR model is also reduced by 25 percent while it has two parameters less than SVJ-IJ-CIR and SVJ-CJ-CIR.
Comparing parameters it is noticeable that the spot volatility V_0 stays nearly invariant under the different stochastic volatility models. Figure 3 shows the time series of V_0.

[Figure 3 here]

Alexander/Nogueira (2005) have explained this feature for "scale–invariant" volatility models. It can be shown that all stochastic volatility models in this paper are scale–invariant. The invariant spot volatility offers a possibility to reduce the estimating effort: after estimating the spot volatility in a scale–invariant model, for example in SV-CIR the parameter $V_0^{\text{SV-CIR}}$ is applied as a constant in a different scale–invariant model, for example in SV4. The number of parameters is reduced by one, for example a new model SV3 is created. The performance of SV3 is discussed at the end of the next section.

Table 4 Mean values of the in–sample pricing errors of the overall period

	# parameters	RMSE	APE[%]	AAE	ARPE[%]
CV	1	27.2	6.4	21.8	33.4
CVJ	4	14.3	3.5	11.6	12.4
SV-CIR	5	5.0	1.1	3.9	7.3
SVJ-CIR	8	3.6	0.8	2.8	5.7
SVJ-CJ-CIR	10	3.4	0.8	2.7	4.5
SVJ-IJ-CIR	10	3.5	0.8	2.6	5.1
SV-OU	5	3.9	0.9	3.0	4.6
SVJ-OU	8	3.2	0.7	2.5	4.9
SV4	4	4.4	1.0	3.4	5.1
SVJ7	7	3.5	0.8	2.7	5.5

3.4 In–sample performance

The objective in the remainder of this section is to analyse the in– and out–of–sample pricing errors that are calculated after each daily calibration: "Root Mean Square Error" (RMSE), "Average Absolute Error" (AAE), "Average Percentage Error" (APE) and "Average Relative Percentage Error" (ARPE) with

$$
\begin{aligned}
RMSE &= \sqrt{\sum_{\text{options}} \frac{(\text{market price} - \text{model price})^2}{\text{number of options}}}, \\
AAE &= \sum_{\text{options}} \frac{|\text{market price} - \text{modelprice}|}{\text{number of options}}, \\
APE &= \frac{1}{\text{average price}} AAE, \\
ARPE &= \frac{1}{\text{number of options}} \sum_{\text{options}} \frac{|\text{market price} - \text{model price}|}{\text{market price}}. \qquad (12)
\end{aligned}
$$

The time series of the pricing error RMSE is given in Figure 4.

[Figure 4 here]

The plot contains the RMSE of the ten relevant option pricing models, but on the first sight there are only three different lines to see. The top line belongs to the CV model, the Black–Scholes benchmark. Under the CV model the pricing error line produced by the CVJ jump–diffusion model is located. The RMSE of CVJ is very close to the RMSE of CV in high volatile periods[h] of the data sample. In less volatile phases the CVJ's pricing errors are close to the stochastic volatility models that present themselves as one line in Figure 4.

The mean values of the pricing errors are listed in Table 4. Summing up the CV model has always the worst in–sample pricing performance. The pure jump–diffusion CVJ model fails especially in high volatile periods of the market. Models that include stochastic volatility match always the market prices very well. So the

[h] In comparison to Figure 2 the high volatile period is around trading day 100 to 300.

Table 5 Performance of SV3

	Trading days 1 to 100	
	RMSE	ARPE[%]
CV	27.8	20.4
SV4	4.5	3.72
SV3 $V_0^{\text{SV3}} = V_0^{\text{SV-CIR}}$	4.7	3.73
SV3 $V_0^{\text{SV3}} = \sigma^{\text{CV}}$	6.3	5.1

level of their pricing errors is independent of the market development. Stochastic volatility is able to generate down movements as well as up movements in the underlying. Thus this component is essential in a reliable option pricing model.
Adding jumps in the underlying reduces in the OU– and in the CIR–models the pricing errors. Within the CIR–models SVJ-IJ-CIR and SVJ-CJ-CIR that model also jumps in volatility have the lowest pricing errors. That is what I expected, because SVJ-IJ-CIR and SVJ-CJ-CIR have the largest number of parameters to fit the market. But the improvement of RMSE is less than six percent. In comparison with the additional costs of two new parameter the improvement is not convincing. In spite of the ARPE of SVJ-IJ-CIR that is more than 21 percent lower than the ARPE of SVJ-CIR.
Looking at the pricing errors of the four OU–models the six CIR–models pale in comparison. SV-OU is the direct competitor to SV-CIR, but the RMSE of SV-OU is around 20 percent less than the RMSE of SV-CIR. The RMSE of SVJ-OU is also around ten percent lower than the RMSE of SVJ-CIR. The SVJ-OU model has the lowest errors in any event except for ARPE and has so the best overall in–sample performance. I conclude that the models using an OU–process to model volatility need less parameters to reach a certain fitting–level. The combination of jumps in the underlying and OU volatility is recommended, because it has a good trade–off of number of parameters and performance.
In Section 3.3 the SV3 model was introduced. The performance of this model over a sample period of the first 100 trading days is listed in Table 5. Replacing the parameter V_0 in SV4 with $V_0^{\text{SV-CIR}}$ of SV-CIR leads to nearly the same pricing error. The RMSE of SV3 is only around three percent worse than the RMSE of SV4. A real advantage in terms of computing time and complexity is to replace the parameter V_0 with the easy to determine parameter σ^{CV} of the CV model. The RMSE of this version of the SV3 model is also shown in Table 5. In comparison with the performance of the original CV model it seems that the SV3 model can handle the parameter σ much better.

3.5 Minimised probabilities for negative values in the OU volatility

In Section 2.3 I have rebutted that the OU volatility is discarded due to negative values in the volatility. Although it is very useful to know how many times a

Table 6 Mean values of RMSE and the corresponding probability of the trading days 100 to 200

	RMSE	Probability [%]
SV-OU	4.87	18
SV-OU, $g = 10^3$	5.70	12
SV-OU, $g = 10^4$	7.35	3
SV-OU, $f = 5$	7.94	1
SV-CIR	5.72	0
SVJ-OU	4.14	13
SVJ-OU, $g = 10^3$	4.63	6
SVJ-OU, $g = 10^4$	5.79	2
SVJ-OU, $f = 5$	6.31	1
SVJ-CIR	5.25	0

negative value occurs. The probability of a negative value is given by[i]

$$P\left(V_t < 0\right) = \phi\left(-\frac{E[V_t]}{\sqrt{\mathrm{Var}[V_t]}}\right), \tag{13}$$

where expectation and variance are written by

$$E[V_t] = e^{-\kappa t}V_0 + (1 - e^{-\kappa t})\theta, \qquad \mathrm{Var}[V_t] = \frac{\beta^2}{2\kappa}\left(1 - e^{-2\kappa t}\right). \tag{14}$$

In order to favour combinations of parameters that have a small probability to become negative over uncontrolled combinations I have joined the objective function of the minimisation problem (8) with a penalty function. Two new constrained objective functions are

$$\begin{aligned} G(X) &= F(X) + g \cdot P\left(V_t < 0 \middle| X\right), \\ G(X) &= F(X) + f \cdot F(X) \cdot P\left(V_t < 0 \middle| X\right), \end{aligned} \tag{15}$$

with parameter vector X, unconstrained objective function F and weighting factors g and f. The weighting factors control how much the penalty function dominates over the original calibration problem. The Figure 5 shows that the probability of negative values goes down close to zero.

[Figure 5 here]

But constraints in a minimisation problem always increase the objective function value. So the pricing errors arise using the objective function (15). Comparing the RMSE in Table 6 it is possible to decrease the probability and achieve lower pricing errors than the reference model SV-CIR simultaneously. Summing up the application of constrained objective functions (15) offers a tool to control the probability for negative values if the user is unfamiliar with negative values in volatility.

[i] see e.g. Schoebel/Zhu (1999)

Table 7 Mean values of the out–of–sample pricing errors of the overall period

	# parameters	$\mathrm{RMSE_{OOS}}$	$\mathrm{AAE_{OOS}}$	$\mathrm{ARPE_{OOS}}[\%]$
CV	1	28.61	22.65	34.62
CVJ	4	17.27	13.80	13.16
SV-CIR	5	10.79	9.24	9.97
SVJ-CIR	8	10.27	8.81	8.90
SVJ-IJ-CIR	10	10.23	8.79	8.41
SVJ-CJ-CIR	10	10.21	8.78	8.38
SV-OU	5	10.40	8.89	7.64
SVJ-OU	8	10.20	8.79	8.32
SV3, $V_0^{SV3} = \sigma^{CV}$	3	14.75	12.14	10.75
SV4	4	10.54	8.97	7.80
SVJ7	7	10.22	8.79	8.32

3.6 Out–of–sample performance

In Section 3.4 it was shown that the in–sample pricing errors of daily calibration decrease if the number of parameters are increased. But one may raise the question whether the added parameters explain more about the market structure or whether they only describe the daily noise. An out–of–sample cross–sectional study can answer this question because in out–of–sample pricing overfitting has a bad influence on the performance of a model.
The out–of–sample study is implemented as follows: the estimated parameters of yesterday, day $t-1$, are used to price options, that have an observable counterpart in the market, today on day t, Pricing errors like in equation (12) are computed for the difference between the model price and the market price. This is done for each trading day for each model.
The mean values of the pricing errors are reported in Table 7. Comparing the errors it seems that there exists for each error measure a lower bound the models converge to. The range between the lowest and the highest error is much smaller than the range of the in–sample errors.
The out–of–sample RMSE of SVJ-IJ-CIR and SVJ-CJ-CIR is less than one percent minor than the RMSE of SVJ-CIR. This indicates overfitting in SVJ-IJ-CIR and SVJ-CJ-CIR.

4 Exotic options and OTC pricing

The objective of this section is to compute prices for several exotic options. The parameters that are used for pricing are obtained by the calibration described in Section 3. The exotic options are Asian options, Forward Start options, Lookback and Barrier options. These options are all path–dependent. The call payoff of each type of exotic option and some characteristics are summarised in this section[j].
All types of Asian options have in common that their payoff contains a mean of the

[j]see e.g. Glasserman (2003)

prices of the underlying. For example with

$$\overline{S_T} = \frac{1}{N} \sum_{i=1}^{N} S(t_i) \tag{16}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$, the payoff of a "fixed" and a "floating" Asian call are given by

$$\begin{aligned} C_T^{\text{fix}} &= \max\left\{\overline{S_T} - K, 0\right\}, \\ C_T^{\text{float}} &= \max\left\{S_T - \overline{S_T}, 0\right\}. \end{aligned} \tag{17}$$

The strike of a Forward Start option is fixed in t_0 as a percentage of the underlying S_1 in $t_1 > t_0$. The option premium has to be paid in t_0 when the true value of the strike is still unknown. The maturity is $t_2 > t_1$.
Lookback options have "fixed" or "floating" strikes like Asian options, but instead of the mean the maximum $S_{\max}$ or the minimum $S_{\min}$ of the underlying asset price during the duration is considered:

$$\begin{aligned} C_T^{\text{fix}} &= \max\{S_{\max} - K, 0\}, \\ C_T^{\text{float}} &= \max\{S_T - S_{\min}, 0\}. \end{aligned} \tag{18}$$

In Figure 6 a fixed Lookback call option with moneyness $K/S_0 = 0.9$ and with maturity $T = 0.5$ is priced by different option pricing models, for example[k].

[Figure 6 here]

The payoff of a Barrier option is the same as the payoff of a plain vanilla option if a prespecified event occurs. If this event does not occur the value of the Barrier option is zero. A "down–and–out" (below: DOB) call is worthless if the underlying drops below a certain barrier during duration. A "up–and–in" call is worthless before the underlying cross the barrier. In the same way "down–and–in" and "up–and–out" (below: UOB) calls are defined. With barrier B, maturity T and strike K the payoff of an "up–and–out" call is given by

$$C_T^{\text{UOB}} = \vec{1}\left\{\tau_B^{\text{Up}} > T\right\} \max\{S_T - K, 0\}, \tag{19}$$

where

$$\tau_B^{\text{Up}} = \inf\{t_i : S_{t_i} < B\},$$

$0 = t_0 < t_1 < \cdots < t_n = T$, $i = 0, \ldots, n$ and $\vec{1}\{.\}$ as the indicator function.
The pricing of European plain vanilla options can be done in a fast and proper way described in Section 3.1. However the pricing of path–dependent options is still a challenging problem. A universal but slow method is Monte Carlo simulation. A lot of facts about Monte Carlo methods in financial engineering are summarised by Glasserman (2003). In order to generate sample path for the Monte Carlo simulation time–discretisations of the underlying models are required. Lord et al. (2006) classified biased simulation schemes for stochastic volatility models. In the following study I have used the "IJK" scheme that was also proposed by Kahl/Jaeckel (2005).

[k]Various option prices for Asian, Forward Start and Barrier options with different moneyness levels and maturities are shown in Ender (2008).

5 Model risk of OTC options

Among other payoff constructions the exotic options of Section 4 are OTC contracts. For OTC options market prices are not available, because each price is unique. Each investor has to decide which model and which price for the exotic option he wants to use and so underwrites model risk.
To deal with derivatives involves a certain number of different risk types. A leading risk is the market risk that includes the interest rate risk and the volatility risk. These risk types have directly influence on option prices. Although credit risk of listed derivatives is nearly eliminated because of securities, OTC options can contain credit risk. Additional risk types are operational risk and liquidity risk. All kinds of bugs and failures in the IT system are subsumed under operational risk. Liquidity risk appears if a trader can not buy his optimal hedging portfolio because of market frictions.
In addition to these risk types there is always model risk, if a financial model is used because of simplified assumptions of the reality. In this study model risk is the risk component that still remains in the model even if every part of option pricing is done with preciseness as programming, hedging and valuation.
To manage a risk in addition to a qualitative description it is necessary to measure the risk quantitatively. Cont (2004) derived a coherent risk measure for model risk. He proposed to subtract the absolute lowest price $\underline{\pi}(X)$ for the exotic option X from the absolute highest price $\overline{\pi}(X)$ in a set of option pricing models $\mathcal{Q}$ to get a monetary quantity of model risk:

$$\mu_{\mathcal{Q}}(X) := \overline{\pi} - \underline{\pi}. \tag{20}$$

However, the problem with the measure $\mu_{\mathcal{Q}}(X)$ is that the absolute prices, especially $\overline{\pi}$ and $\underline{\pi}$, contain the fitting errors discussed in Section 3.4. To take the bias out of the exotic option prices it is important to normalise the absolute prices. The normalisation of the exotic option uses a price of an appropriate plain–vanilla option with the same strike priced in each model. So the fluctuations because of the fitting errors can be compensated and eliminated. An example of the difference between the monetary price (see Figure 6) and the normalised price of a Lookback option is given in Figure 7.

[Figure 7 here]

In order to use a model risk measure for a set of models a normalisation of the risk measure in equation (20) should be done by the mean of the normalised prices of all models. Subtracting now the lowest from the highest price results no longer in a monetary quantity but in a percental ratio in terms of the average normalised price. The model risk measure $\overline{\mu_{\mathcal{Q}}}$ I propose is an extension of the Cont (2004) measure in equation (20) and is defined in equation (21):

$$\overline{\mu_{\mathcal{Q}}}(X) := \frac{\mu_{\mathcal{Q}}(X)}{\frac{1}{m}\sum_{k=1}^{m} C_k^{\text{norm}}(X)}. \tag{21}$$

The question to answer first is which exotic options is more vulnerable to model risk than other exotic options. Due to normalisation this is possible with the model risk measure (21). In Figure 8, 9 and in Figure 10 the model risk of Forward Start, fixed Asian and fixed Lookback call options on each trading day from 2002 to September

Table 8 The effect of an increasing maturity on the model risk depending on exotic option types

Model risk effects increasing maturity	Options in–the–money	at–the–money	out–of–the–money
Forward Start	↗	↘	↘
fixed Strike Asian	↗	↗	↘
floating Strike Asian	⟶		
fixed Strike Lookback	↗	↗	↘
floating Strike Lookback	⟶		

2005 are plotted. Two different maturity structures and three different moneyness levels $k_i = K_i/S_0$ are shown.

[Figure 8 here]

[Figure 9 here]

[Figure 10 here]

Model risk of these option types, Forward Start, fixed Asian and fixed Lookback options, increases if moneyness is risen. Model risk of long term options that are out–of–the–money is less than the model risk of short term options. On the other hand in each case model risk of long term options that are in–the–money is higher than the model risk of short term options. For options at–the–money the results are different what is reported in Table 8. Forward Start, Asian and Lookback options have a rising major model risk when the market is nearly trendless during the days around 700 to 850 at the end of 2004 and the beginning of 2005.
Among moneyness and maturity model risk of Barrier options is also determined by the level of the barrier. The farther the barrier is away from the spot, the less is the model risk and the other way round. These effects and additional market effects of DOB and UOB are shown in Figure 11.

[Figure 11 here]

6 Conclusion

The main findings of this paper are summarised in this section. The benchmark CV model of Black and Scholes performs poorly in– and out–of–sample. However alternative models including stochastic volatility perform much better.
To model volatility using an OU–process has advantages over the more in common CIR–process. Negative values in volatility are manageable and they are not hindering.
Adding jumps to the volatility process is in theory a very good tool to explain market behaviour. But only little improvements to the in–sample fitting and hardly any improvements to the out–of–sample performance are not convincing empirically.
Asian options have a lower model risk than Forward Start, Lookback and Barrier options.

Model risk measured by an new extension of the model risk measure of Cont (2004) increases with the moneyness. The maturity effects depend on whether the option is in–, at– or out–of–the–money. Model risk could become a very important risk factor for out–of–the–money options in a trendless, slightly bullish market. In this case a very precisely pricing is needed to avoid model risk. The examples of OTC options show how important it is to know how sensitive a certain exotic option is to model risk.
Summing up there are urgent needs of quantitative models for option pricing and hedging strategies. The development of more precisely risk measures for model risk is a required issue of further research.

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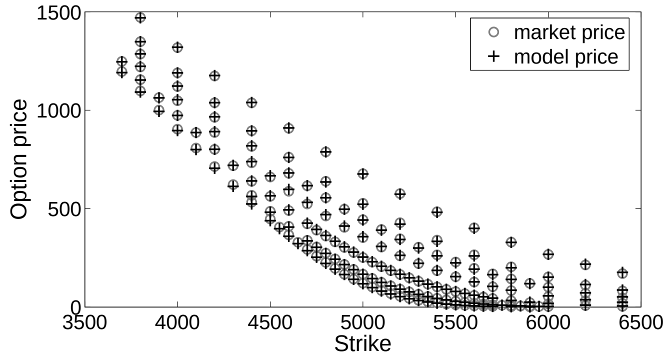


Figure 1 Calibration example for showcase trading day: "o" market prices, "+" corresponding model prices determined by calibration

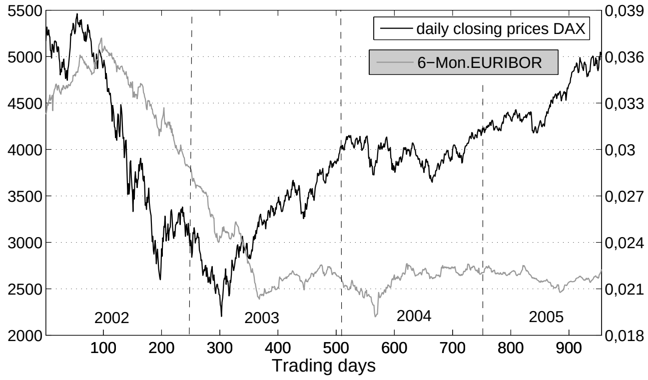


Figure 2 DAX30 and EURIBOR interest rates, 01. January 2002 to 30. September 2005

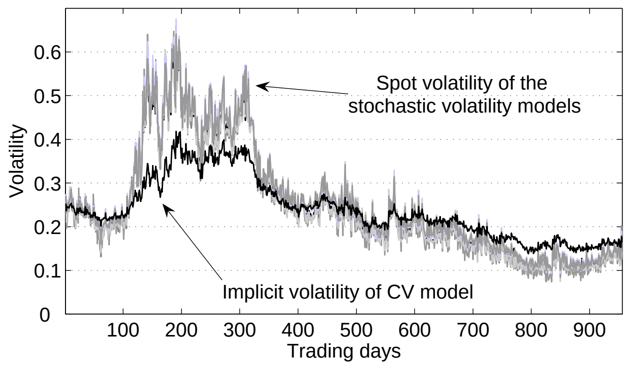


Figure 3 Spot volatility V_0 of different scale–invariant stochastic volatility models and the implicit volatility, 01. January 2002 to 30. September 2005

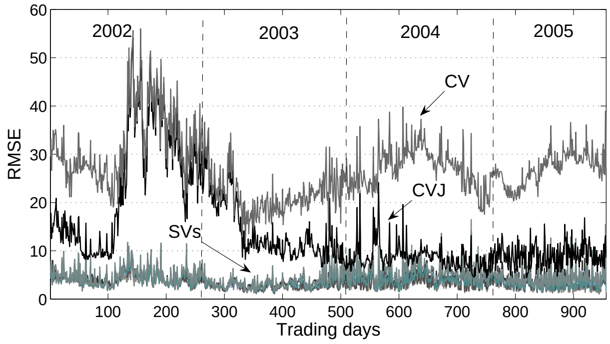


Figure 4 RMSE time series of the ten option pricing models, 01. January 2002 to 30. September 2005

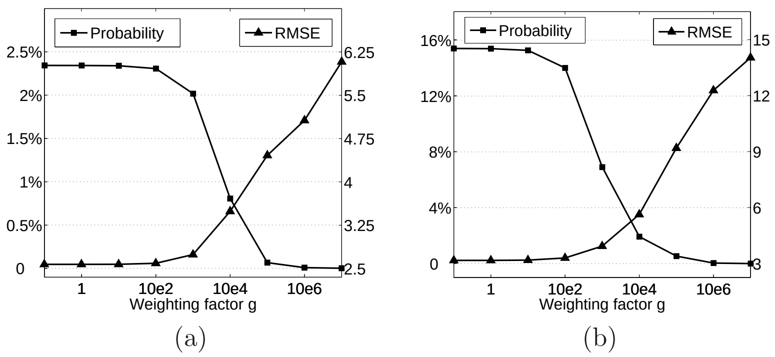


Figure 5 SV-OU: probability and RMSE for several weighting factors g on the trading days (a) 101 and (b) 198

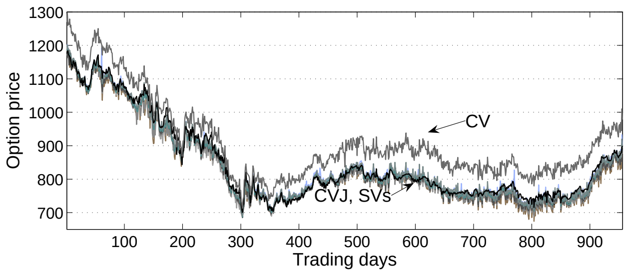


Figure 6 Absolute prices of a fixed Lookback call option with $k = 0.9$ with maturity $T = 0.5$, 01. January 2002 to 30. September 2005

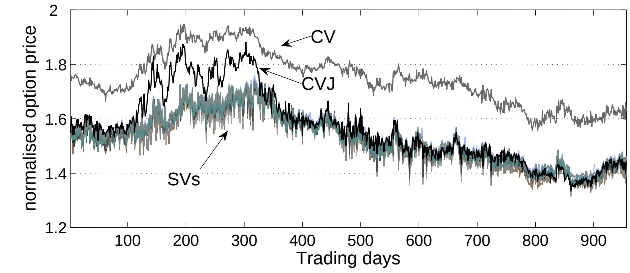


Figure 7 Normalised prices of a fixed Lookback call option with $k = 0.9$ with maturity $T = 0.5$, 01. January 2002 to 30. September 2005

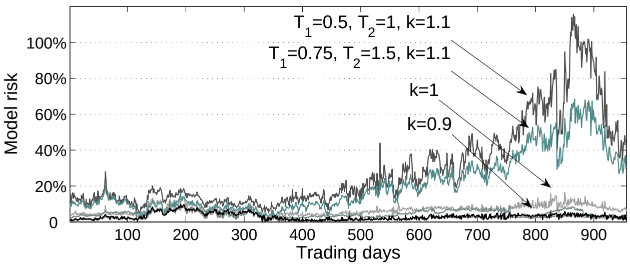


Figure 8 Model risk in a percental ratio in terms of the average normalised price of Forward Start call options with $k = 0.9$, $k = 1$, $k = 1.1$ with maturity $T_1 = 0.5$, $T_2 = 1$ and $T_1 = 0.75$, $T_2 = 1.5$, 01. January 2002 to 30. September 2005

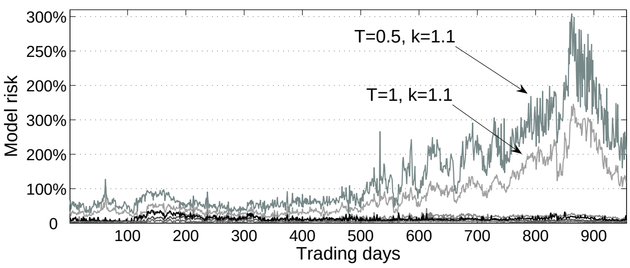


Figure 9 Model risk in a percental ratio in terms of the average normalised price of fixed Strike Asian call options with $k = 0.9$, $k = 1$, $k = 1.1$ with maturity $T = 0.5$ and $T = 1$, 01. January 2002 to 30. September 2005

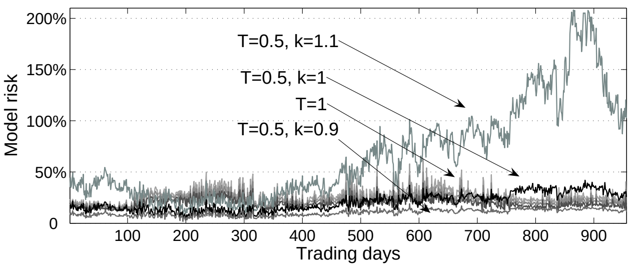
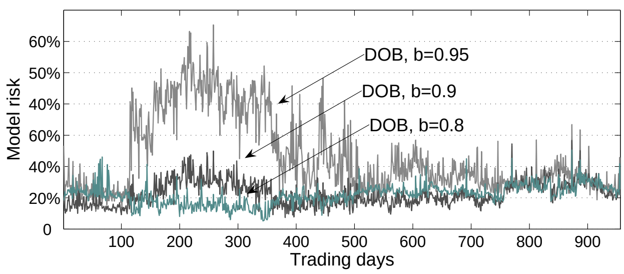
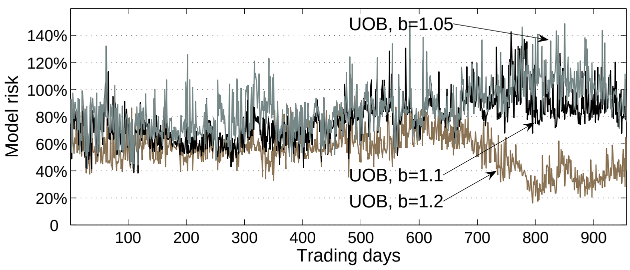


Figure 10 Model risk in a percental ratio in terms of the average normalised price of fixed Strike Lookback call options with $k = 0.9$, $k = 1$, $k = 1.1$ with maturity $T = 0.5$ and $T = 1$, 01. January 2002 to 30. September 2005



(a)



(b)

Figure 11 Model risk in a percental ratio in terms of the average normalised price (a) DOB call options with barriers $b = 0.8$, $b = 0.9$, $b = 0.95$ and (b) UOB call options with barriers $b = 1.05$, $b = 1.1$, $b = 1.2$ with maturity $T = 0.5$, 01. January 2002 to 30. September 2005