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The Importance of Estimation Uncertainty in a Multi-Rating Class Loan Portfolio

Abstract

This article seeks to make an assessment of estimation uncertainty in a multi-rating class loan portfolio. Relationships are established between estimation uncertainty and parameters such as probability of default, intra- and inter-rating class correlation, degree of inhomogeneity, number of rating classes used, number of debtors and number of historical periods used for parameter estimations. In addition, by using an exemplary portfolio based on Moody's ratings, it becomes clear that estimation uncertainty does indeed have an effect on interest rates.

Keywords: credit portfolio risk, estimation uncertainty, bootstrapping, economic equity

JEL Classification: C15, D81, G11

Die Bedeutung von Schätzunsicherheit im hinsichtlich der Bonität inhomogenen Kreditportfolio

Zusammenfassung

Der Beitrag beschäftigt sich mit der Bewertung von Schätzunsicherheit in einem hinsichtlich der Bonität inhomogenen Kreditportfolio. Es wird zunächst gezeigt, dass neben dem in der Literatur bereits diskutierten Zusammenhang zwischen der Schätzunsicherheit und der Anzahl historisch verfügbarer Perioden beziehungsweise der Ratingklassengröße auch ein Zusammenhang zwischen diesem Modellrisiko und der Bonität, dem Grad der Inhomogenität, der Innerklassen- und Interklassenkorrelation sowie der Ratingklassenzahl besteht. Darüber hinaus wird am Beispiel eines auf Moody's-Ratings beruhenden Portfolios verdeutlicht, dass durch eine Berücksichtigung dieses Modellrisikos der Kreditzins in relevantem Umfang steigen kann.

Schlagwörter: Kreditrisikobewertung, Schätzunsicherheit, Bootstrapping, ökonomisches Eigenkapital

JEL-Klassifikation: C15, D81, G11

The Importance of Estimation Uncertainty in a Multi-Rating Class Loan Portfolio

1 Introduction

An accurate assessment of risk calls for the correct parameterization of the underlying risk models. If risk models are parameterized incorrectly, the risk assessment will also be inexact. The consequences of such inaccuracies could be flawed management decisions, in particular the inadequate provision of equity as a risk buffer. Jorion (1996) points to this problem in the context of market risks. Because model assumptions and parameter estimations are used for Value at Risk calculations he notes that a Value at Risk (VaR) is not an incontrovertible account. This means that VaR calculation is affected by model risks which must be taken into account.[1]

There are two kinds of model risk. On one hand, there is the danger that a manager may use a false or incorrectly parameterized model. Such a case would exist if, for example, the stochastic processes which are used in a model do not correspond to the actual processes. On the other hand, the parameters of risk models are unknown and must therefore be estimated. However, because of estimation errors, this parameter estimation may also be a source of model risk. It is thus possible that incorrect estimation techniques are used, or that problems may occur with outliers. Yet another problem may arise from estimation uncertainty, also referred to as estimation noise.[2] If the estimate is based on historical data, it must be taken into account that historical observations are only random realizations of the unknown risk distribution. Thus the danger exists that a risk model is incorrectly parameterized because of randomly atypical historical data. For example, we may observe “heads” 60 times after 100 coin tosses. A risk manager who does not know that both sides of the coin have the same probability in this game would estimate the probability of heads as 60% because of this atypical data history. In such cases, the parameter estimation itself is a risk factor which must be incorporated in risk assessment. This means that a company requires extra economic equity because of possibly atypical historical realizations of the underlying unknown risk distribution.

The focus of this paper is on such estimation uncertainty in the modeling of credit risks. Using a Bernoulli mixture model, the paper investigates the impact of estimation uncertainty on the amount of economic equity required by a creditor. An important advantage of the credit risk model used is that the estimation of both the probabilities of default

1 *Jorion* (1996), pp. 47 ff., see also *Dowd* (2000), *Christoffersen*, *Goncalves* (2005).

2 *Sibbertsen, Stahl, Luedtke* (2008), pp. 65 ff.; *Crouhy, Galai, Mark* (1998), pp. 273 ff.; *Tarashev, Zhu* (2008), pp. 1249 ff., *Tarashev* (2009), p. 1.

and the correlation coefficients are based on the bank's own credit history. This implies that no capital market data are necessary.

A second aim of this inquiry is to examine the relationships between estimation uncertainty and risk model parameters, such as probability of default, intra and inter rating class correlation, degree of inhomogeneity, number of rating categories used, and the number of debtors and of historical periods used for parameter estimations. The knowledge of such relationships assists in the evaluation of the relevance of estimation uncertainty in a credit portfolio, enhancing, for example, the ability of regulators to identify banks with high model risks. Unfortunately, because of the relationships between estimation uncertainty and parameters of the risk model no general statement about the necessary amount of extra economic capital can be made, making it impossible to say that in general a bank needs 20% more equity to cover the model risk "estimation uncertainty". However, in order to provide an idea of the importance of this model risk this article describes the effect of estimation uncertainty on the equity requirements and interest rates of a bank using a portfolio based on Moody's ratings.

The remainder of the paper is organized as follows. Section 2 focuses on model risk quantification. Section 3 describes the model and the methods used for its parameterization. In Section 4 it is explained how estimation uncertainty can be quantified by bootstrapping. Section 5 discusses a simulation study where relationships between portfolio parameters and estimation uncertainty are investigated. In order to provide some idea of the relevance of this model risk, an example based on Moody's ratings is provided in Section 6. Section 7 concludes the paper.

2 Model Risk in the Literature

Sibbertsen, Stahl and Luedtke (2008) distinguish two approaches for measuring model risks. On one hand, using the Bayesian approach, a risk can be measured by different models. Hence different key figures such as Value at Risk can be calculated. The overall risk can then be calculated as the (weighted) average of these key figures. Applied to the problem of estimation uncertainty, this approach requires that model parameters themselves are modeled by random variables, i.e. by probability distributions. On the other hand, model risks can be measured by a so-called worst case approach. In this approach, the risk is calculated by different models and the gap between the model which calculated the highest risk (worst case model) and the nominal model is called the model risk.[3]

The worst case approach is very conservative. If one applies this approach to credit risk management it is necessary to build worst case scenarios for model parameters like Probability of Default (*PD*) or correlation coefficients. A practical way to do this is to

3 *Kerkhof, Melenberg, Schumacher* (2002), pp. 11 ff., *Sibbertsen, Stahl, Luedtke* (2008), pp. 65 ff.

use the upper bounds of confidence intervals. The use of such upper interval bounds to assess estimation uncertainty is very common in credit risk assessment, especially in the estimation of probabilities of default.[4] Such a procedure is used by *Lawrenz* (2008), for example. He applies the interval bounds of probability of default to calculate the risk weights of the Basel II IRB approach. In doing so, he calculates the range of these weights.[5] *Rösch* (2004) also suggests a correction of the estimated probability of default by the upper bounds of confidence intervals, although he does point out that this practice is very conservative.[6] The disadvantage of such an approach is the double definition of confidence levels, on the one hand, at the definition of upper bounds of the confidence interval and on the other hand at the definition of confidence level of Value at Risk. In this way the equity requirement is overestimated. In addition, no one knows the real confidence level of Value at Risk when calculated using such an approach. For example, if someone constructs a credit risk distribution using the 95% upper bound of a *PD*-estimator, at which confidence level should he define the Value at Risk so that, with a probability of 99.9%, all credit losses will be covered? It is apparent that it should not be at the 99.9% level because he has already used an unlikely value for the model parameter. This problem becomes even more relevant when several parameters must be estimated and the upper limit of a confidence interval is used for each parameter.

An alternative to the worst case approach is the Bayesian approach. The challenge of this procedural method is the need to quantify the probabilities of diverse scenarios.[7] In other words, for every possible manifestation of an estimated parameter a probability must be attributed. Therefore it is necessary to describe estimators by random variables respectively to model the estimated parameters themselves by probability distributions. However, if the parameters are modeled by distributions, analytical solutions typically will be impossible. The Bayesian approach thus implies the considerable disadvantage of high computational effort in assessing risk.

Those in the literature who have followed the Bayesian approach to assess credit risk include *Tarashev* (2009), *Löffler* (2003), *Gössl* (2005), *Hamerle and Rösch* (2006) and *Dannenberg* (2010). These authors use probability distributions to model parameters. *Löffler* (2003) and *Dannenberg* (2010), for example, establish the distribution of a failure rate using bootstrapping.[8] *Hamerle and Rösch* (2006) assume a normal distribution, but they point out that this assumption may be inappropriate in small samples.[9] *Tara-*

4 *Cassart* et al. (2007); *Christensen* et al. (2004); *Dannenberg* (2008); *Dannenberg* (2010); *Hamerle* et al. (2005); *Hanson, Schuermann* (2006); *Höse*, *Huschens* (2003); *Höse* (2007); *Pluto, Tasche* (2005); *Stein* (2003); *Trück, Rachev* (2005).

5 *Lawrenz* (2008), pp. 231 ff.

6 *Rösch* (2004), pp. 20 ff.

7 *Sibbertsen, Stahl, Luedtke* (2008), pp. 72 ff.

8 *Löffler* (2003), pp. 1431 ff.

9 *Hamerle, Rösch* (2006), pp. 116 ff.

shev (2009), for example, uses a beta probability distribution for modeling correlation.[10] In order to generate the joint probability distribution of the estimated model parameters, the bootstrap approach is used here.

But how can risk be calculated, if model risks are to be taken into account? *Löffler* (2003) suggests mixing the probability distribution of the estimator and the probability distribution of the original risk model.[11] Tarashev (2009) describes a similar procedure.[12] The following example should illustrate this approach.

Assume that the credit risk of a company can simply be modeled by a binomial distribution *Bin*(*PD*, *N*) with Probability of Default *PD* and number of debtors *N*. If $N = 500$ and $PD = 10\%$, the $VaR_{99\%}$ will correspond to 66 credit defaults. But what would happen if *PD* is uncertain? Assume that *PD* could also be 8% or 12% (each with a probability of 20%) and 10% with a probability of 60%. Then three binomial distributions could be parameterized and hence three $VaR_{99\%}$ (55, 66 and 77 defaults) could be calculated. Would one of these three values correspond with the "true" $VaR_{99\%}$? Probably not. In order to calculate the "true" $VaR_{99\%}$ the uncertainty about parameter *PD* must be incorporated. Modeling *PD* by a probability distribution $PD \sim \Upsilon$ and mixing this distribution with the original credit risk distribution *Bin*(*PD*, *N*) will lead to a new credit risk distribution $Bin(\Upsilon, N)$ which can be used to calculate the "true" $VaR_{99\%}$. In this example the "true" $VaR_{99\%}$ is 72 credit defaults.

In this paper, the Bayesian approach is used to model estimation uncertainty. In the following the notation "VaR with estimation uncertainty" will be used to describe the case where the model parameters are random numbers. If only point estimators are used to parameterize a risk model, the notation "VaR without estimation uncertainty" will apply.

3 Model

Owing to the Basel II IRB Approach, the one-factor model (also known as the Vasicek-model):

$$B_i = \sqrt{\rho_{Probit}^{Asset}} F + \sqrt{1 - \rho_{Probit}^{Asset}} U_i \qquad (1)$$

is well known. In this model, both the systematic risk factor $F \sim \Phi(.)$ and the unsystematic risk factor $U_i \sim \Phi(.)$ are assumed to be normally distributed. These factors are mutually independent. The random variable B_i can be interpreted as the return on a firm's assets. The coefficient ρ_{Probit}^{Asset} is known as the "asset correlation". A debtor *i* will default if the return on a firm's assets falls below a threshold c_i. If it is possible to derive

10 *Tarashev* (2009), p. 10.

11 *Löffler* (2003), pp. 1448 f.

12 *Tarashev* (2009), pp. 10 f.

the probability of default from market or historical data, the threshold c_i can be calculated by the inverse value of the Gaussian distribution at PD_i: $c_i = \Phi^{-1}(PD_i)$. In a homogeneous credit portfolio, a portfolio of borrowers with identical credit ratings, PD_i, corresponds to the expected probability of default PD of all debtors. Conditional on the realization f of the systematic risk factor F the conditional probability of default $\tilde{\pi}(F=f)=\pi$ of borrowers can be calculated as:

$$
\begin{aligned}
&B_i = \sqrt{\rho_{Probit}^{Asset}}F + \sqrt{1-\rho_{Probit}^{Asset}}U_i \le c_i \\
&\sqrt{\rho_{Probit}^{Asset}}F + \sqrt{1-\rho_{Probit}^{Asset}}U_i \le \Phi^{-1}(PD) \\
&U_i \le \frac{\Phi^{-1}(PD)-\sqrt{\rho_{Probit}^{Asset}}F}{\sqrt{1-\rho_{Probit}^{Asset}}} \\
&\rightarrow \tilde{\pi}(F=f)=\pi=\Pr\left(U_i \le \frac{\Phi^{-1}(PD)-\sqrt{\rho_{Probit}^{Asset}}F}{\sqrt{1-\rho_{Probit}^{Asset}}} \Big| F=f\right)=\Phi\left(\frac{\Phi^{-1}(PD)-\sqrt{\rho_{Probit}^{Asset}}f}{\sqrt{1-\rho_{Probit}^{Asset}}}\right).
\end{aligned}
\tag{2}
$$

The one-factor model can be transformed into a credit risk model in which the factors need not be modeled explicitly. The so-called Bernoulli mixture models use a probability distribution to model the conditional probability of default. In practice, the probability distributions most used to model conditional probability of default are the Beta, Logit and Probit distributions. It is assumed that the conditional probability of default is described by a stochastic random variable $\tilde{\pi}$, the possible realizations of which are denoted π. The underlying probability distribution of $\tilde{\pi}$ can then be used to draw possible realizations of this random variable. For example, in the one-factor model described above, the conditional probability of default can be modeled by a Probit distribution: $\tilde{\pi}=\Phi(\mu+\sigma F)$, with $\mu = \frac{\Phi^{-1}(PD)}{\sqrt{1-\rho_{Probit}^{Asset}}}$ and $\sigma^2 = \frac{\rho_{Probit}^{Asset}}{1-\rho_{Probit}^{Asset}}$.[13] This means that in this case the probability distribution of conditional probability is parameterized by the average probability of default (PD) and the so-called asset correlation ρ_{Probit}^{Asset}. The underlying cumulative distribution function of the mixing variable is:[14]

$$
F_{\tilde{\pi}}(\pi)=\Phi\left(\frac{\sqrt{1-\rho_{Probit}^{Asset}}\,\Phi^{-1}(\pi)-\Phi^{-1}(PD)}{\sqrt{\rho_{Probit}^{Asset}}}\right). \tag{3}
$$

If this function is mixed with a binomial distribution, the probability of a specific number of defaults $H = h$ in a homogeneous loan portfolio with the size N can be calculated by:[15]

13 *McNeil, Frey, Embrechts* (2005), p. 361.

14 *Höse* (2007), p. 52.

15 *McNeil, Frey, Embrechts* (2005), p. 354.

$$\Pr\left(H=h\right)=\binom{N}{h}\int_0^1 \pi^h\cdot(1-\pi)^{N-h}\,dF_{\tilde{\pi}}\left(\pi\right). \qquad (4)$$

In cases of inhomogeneous loan portfolios, r ($r = 1, 2,\ldots, R$), groups with different credit ratings can be built. It is assumed that all borrowers in such a group r have the same rating. The size of a rating group at time t is N_t^r. In an inhomogeneous loan portfolio the conditional probabilities of default must be modeled by a multivariate distribution $F_{\{\tilde{\pi}_t^r\}}\left(\{\pi_t^r\}_{tr}\right)$ with marginal totals $F_{\tilde{\pi}_t^r}\left(\pi_t^r\right)$. As indicated above, the marginal totals are modeled by Probit distributions which are parameterized by the average probability of default PD^r and the asset correlation $\rho_{Probit}^{r,Asset}$. In this paper the Asset correlation $\rho_{Probit}^{r,Asset}$ will also be denoted as intra rating class correlation.

But a correlation exists not only between the borrowers of a rating class; correlations between different rating classes are also possible. The correlation ∂^{qw} between two rating classes q and w ($q = 1,\ldots, R$, $w = 1,\ldots, R$) is denominated as an inter rating class correlation. In order to model such dependencies between rating classes, a Gaussian Copula is used here to build the joint distribution of conditional probabilities of default. The multivariate distribution is:[16]

$$F_{\{\tilde{\pi}_t^r\}}\left(\pi_t^1,\pi_t^2,\ldots,\pi_t^R\right)=\mathbf{\Phi_K}\left(\Phi^{-1}\left(F_{\tilde{\pi}_t^1}\left(\pi_t^1\right)\right),\Phi^{-1}\left(F_{\tilde{\pi}_t^2}\left(\pi_t^2\right)\right),\ldots,\Phi^{-1}\left(F_{\tilde{\pi}_t^R}\left(\pi_t^R\right)\right)\right), \qquad (5)$$

where $\mathbf{\Phi_K}(\ )$ denotes a multivariate Gaussian distribution with a symmetrical, positive definite and time-invariant correlation matrix **K** with diagonal elements diag(**K**) = 1 and non diagonal elements ∂^{qw}. Φ^{-1} is the inverse of a standard normal distribution.

An important assumption of this model is that all parameters are time invariant. One could argue that this assumption is unrealistic because of autocorrelation of systematic risk factors. Probability of default should thus also be autocorrelated. However, although the model used is derived from a one-factor model, it is not itself a factor model. The Bernoulli mixture model depends on rating classes. Typically, a rating analyst includes predictions of systematic macroeconomic risk factors in his rating. For this reason, a borrower will presumably receive a worse rating if the analyst predicts unfavourable manifestations of macroeconomic factors, than if he predicts favourable manifestations of these factors. The perfect rating analyst would therefore match borrowers to rating classes in such a manner that the actual probability of default of a rating class corresponded to the long term probability of default of this rating class. In this case, the parameter $\rho_{Probit}^{r,Asset}$ would be very small. However, because rating analysts in the real world are not perfect, not all relevant systematic risk factors are correctly predicted. Therefore, the asset correlation depends on the level of forecast errors. Someone who argues that probabilities of default are autocorrelated therefore assumes a systematic prediction error of rating analysts. Such an assumption is not persuasive.

16 *Cherubini, Luciano, Vecchiato* (2004), p. 147; *Frey, McNeil* (2003), pp. 87 ff.

In the credit model presented, only the number of defaults is modeled. The credit risk parameters Loss given Default (*LGD*) and Exposure at Default (*EAD*) are assumed to be one. This simplification reduces computation efforts in the following sections. Of course, a correlation between *LGD* and *PD* or exposure concentration also influences credit risk. Hence, in a real credit risk assessment such relationships have to be taken into account. Incorporating such risk parameters would certainly be interesting because when more parameters are estimated, more sources of estimation uncertainty can be considered. But in doing so, the paper would become rather complex. Hence, for purposes of simplification, the focus here is on simulation of defaults. One may argue that by including these parameters, the effects which are shown here could be compensated for. This implies that one of the model elements examined here could widen the probability distribution of an estimated parameter and simultaneously constrict the probability distribution of another which has not been considered here. But which parameter could have such an effect? *LGD* estimation typically depends on historical *LGDs* and not on historical default rates which are used here to estimate the model parameters. The same applies for *EAD*. There is thus no rational reason to assume that by including *LGD* and *EAD* the insights of this paper would be obscured.

The equity requirement (*ER*) calculation depends on Value at Risk (VaR_α) and expected loss (*EL*) of loss distribution, where α is the confidence level of Value at Risk. Banks would typically like to make profits. Hence a return on equity (τ) is planned. The proposed profit reduces equity requirements. The equity requirement can be calculated by:

$$ER = \frac{VaR_\alpha - EL}{1+\tau}. \tag{6}$$

The correlation coefficient $\rho_{Probit}^{r,Asset}$ and the expected probability of default PD^r of each rating class must be estimated in order to parameterize the model. The inter class correlation matrix **K** with elements ∂^{qw} must also be estimated. Here two methods of moments, (M1) and (M2), and one maximum likelihood method (ML) are used to estimate the model parameters. The methods are described in Appendix 1.

In the case of maximum likelihood estimation, the two-step canonical maximum likelihood method is chosen. First, the parameters of marginal totals are estimated. Given the parameterized marginal totals, the correlation matrix **K** of the copula is estimated. This method is chosen because of its relatively short computing time compared to alternative ML methods. For the purposes of the simulation study presented here, tens of thousands of credit histories were analyzed; computing time was thus a very important factor. Although this method is not particularly time consuming compared to other ML methods, it nevertheless requires considerable time. For this reason it was used only in sections 5.2, 5.3, 5.4 and 6. In addition to the ML method, all research questions in this paper have been analyzed based on two methods of moment (M1 and M2). The significant advantage of these methods is their short computing time. A disadvantage is the assumption of a time invariant rating class size of method M1. For this reason it will not be possible to investigate whether variance in rating class size causes estimation uncertainty. It must be emphasized at this point that the three methods are used to check for robustness. It is also clear that if three methods lead to three different credit risk assessments, the choice of method itself becomes a source of model risk. But this kind of model risk is not the object of this study.

4 Assessment of Estimation Uncertainty using Bootstrapping

As was explained in section 2, estimation uncertainty can be included in credit risk assessment by modeling the parameter itself by a multivariate probability distribution. Such a probability distribution has several dimensions. For example, in a three-rating class case there are twelve parameters which must be estimated. The multivariate probability distribution thus has twelve dimensions. An analytical derivation of such probability distributions is an unrealistic approach. In cases such as this the bootstrap approach is used for derivation. There are two basic bootstrap approaches, the parametric and the none-parametric approach.[17] In this study, the parametric approach is followed. This method presupposes the existence of an idea of the underlying data generating process of historical observations. This process corresponds to the credit risk model described in section 2. This means that the historical observations are generated by a probability distribution which itself is a mix of a multivariate Probit distribution and R binomial distributions. The following six steps are necessary in the derivation of a parameter probability distribution.

Generate the parameter distribution:

(S1) Based on historical observations, the parameters $PD^r, \rho_{Probit}^{r,Asset}, \mathbf{K}$ ($r = 1,\ldots,R$) are estimated.

(S2) Based on (S1) random numbers of $F_{\tilde{\pi}_t^r}\left(\pi_t^r\right)$ ($r = 1,\ldots, R$ and t = 1,…, T) are drawn from $\mathbf{\Phi_K}(\;)$.

(S3) By inserting the random numbers of the previous step in equation (3) and rearranging to π_t^r for every historical point of time t and for every rating class r ($r = 1,\ldots, R$ and t = 1,…, T), random realizations of conditional probability of default can be generated. The expected probability of default (PD^r) and the asset-correlation $\rho_{Probit}^{r,Asset}$ in equation (3) correspond to the estimated values in step (S1).

(S4) Inserting $R \cdot T$ conditional probabilities of default π_t^r from step (S3) in $R \cdot T$ binomial distributions $Bin(N_t^r;\pi_t^r)$. N_t^r denotes the historical number of debtors at time t in rating class r. Random drawing of a number of loan defaults and computing of default rates for every historical point of time t and for every rating class r ($r = 1,\ldots,R$ and t = 1,…, T).

[17] *Vose* (2005), pp. 181 ff.; *Chernick* (2008), pp. 120 ff.

(S5) Estimation of $PD^r, \rho_{Probit}^{r,Asset}, \mathbf{K}$ on the basis of default rates which were generated in Step (S4). Saving of the estimated parameters as a possible parameter combination.

(S6) Repeat Steps (S2) to (S5) x-times. Unless otherwise specified, x will be 1,000.

By running steps (S1) to (S6) a sample of x possible parameter combinations is generated. Using this combination of parameters, a multivariate parameter probability distribution can be fitted. By inserting this parameter distribution in the credit risk model, estimation uncertainty can be considered for *VaR* calculations. In practice it may not actually be necessary to fit a multivariate parameter probability distribution to simulated data and to draw parameters from this distribution because the parameter combinations of step (S5) can be used directly. A practical way to insert parameter probability distribution in a credit risk model is indicated in steps (S7) to (S12) below.

Building a credit risk probability distribution:

(S7) Random selection of a parameter combination $PD^r, \rho_{Probit}^{r,Asset}, \mathbf{K}$ from the bootstrap sample if estimation uncertainty is considered. If estimation uncertainty is not considered in credit risk assessment, only the parameters originally estimated in step (S1) are used.

(S8) Insertion of the selected parameters in the Gaussian copula $\mathbf{\Phi}_{\mathbf{K}}(\ )$ and random drawing of R realizations of $F_{\tilde{\pi}_{T+1}^r}\left(\pi_{T+1}^r\right)$.

(S9) By inserting the realizations of the previous step in equation (3) and rearranging to π_{T+1}^r the conditional probability of default of forecast period $T + 1$ can be determined. The expected probability of default (PD^r) and the asset-correlation $\rho_{Probit}^{r,Asset}$ in equation (3) corresponds to the selected values in step (S7).

(S10) Insertion of N_{T+1}^r and π_{T+1}^r in binomial distributions $Bin(N_{T+1}^r;\pi_{T+1}^r)$ ($r = 1,\ldots,R$) and drawing of random numbers of loan defaults from these probability distributions for each rating class.

(S11) Adding the defaults of all rating classes and saving this sum.

(S12) Repeating steps (S7) to (S11) y times. Unless otherwise specified, y will be 100,000.

In the following sections, the effect on economic equity of replacing the point estimators with parameter probability distributions will be investigated. The focus is on the question of what relationships exist between estimated parameters or given credit parameters such as portfolio size and the amount of additionally used equity caused by estimation uncertainty.

5 Simulation Study

5.1 Simulation Design

In the following investigations, the required amount of economic equity is calculated twice, once taking estimation uncertainty into account, and once ignoring it. In this way two Value at Risks and hence two amounts of economic equity based on equation (6) are calculated. In the following, VaR_α^{EU} denotes the Value at Risk in the calculation of which estimation uncertainty is considered. The Value at Risk without estimation uncertainty is represented as VaR_α^{o}. The percentage differences between amounts of economic equity $\Delta ER(\%)$ will be investigated in the following section. Hence, in this research study the return on equity (τ) can be ignored.[18]

The credit risk in the credit portfolio model examined here is determined by expected probability of default, inter and intra rating class correlations and number of rating classes. Size of rating classes and number of available historical periods are also relevant for parameter estimation. Whether changing one of these parameters also changes the relevance of estimation uncertainty will be investigated, particularly if increasing a parameter also increases the need for economic capital. For each studied parameter, 50 historical credit portfolios are randomly generated by Monte Carlo Simulation. The Values at Risk $VaR_\alpha^{o}, VaR_\alpha^{EU}$ and the expected loss (EL) are calculated for each generated credit portfolio. In this way, 50 percentage differences between amounts of economic equity $\Delta ER(\%)$ can be calculated. Based on these 50 $\Delta ER(\%)$ it can be determined whether increasing a parameter also leads to an increased or decreased demand for economic equity.

In the following, with the exception of sections 5.4, 5.5, 5.6 and 6, credit portfolios with three rating classes are investigated. It is not necessary here to estimate the parameters in step (S1). This means that the initial values $PD^r, \rho_{Probit}^{r,Asset}, N^r$ are given directly or are randomly drawn from a uniform probability distribution with given upper and lower limits. The rating class size in each case is assumed to be time invariant $N^r = N_t^r$ with $t = 1,\ldots, T$. The limits of the given uniform probability distribution are orientated according to the parameters of Moody's example which is discussed in section 6. In addition, the asset correlation fluctuates only within the limits of Basel II equity requirements, therefore between 0.08 and 0.24.[19] The initial correlation matrix **K** and the number of historical periods T are given deterministically.

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18 $$\Delta ER(\%) = \frac{\frac{VaR_\alpha^{EU} - EL}{1+\tau} - \frac{VaR_\alpha^{o} - EL}{1+\tau}}{\frac{VaR_\alpha^{o} - EL}{1+\tau}} \cdot 100 = \frac{VaR_\alpha^{EU} - VaR_\alpha^{o}}{VaR_\alpha^{o} - EL} \cdot 100.$$

19 *Basel Committee on Banking Supervision* (2005), p. 60.

5.2 Relationship between Estimation Uncertainty and Number of Historical Periods

Initially, the relationship between $\Delta ER(\%)$ and T is investigated. It is expected that increasing T reduces $\Delta ER(\%)$.[20] Here three different time periods T (T = 10, 15 and 20) are studied to simulate three scenarios each with 50 credit portfolios. If the average of $\Delta ER(\%)$ differs in these three scenarios, an effect of T on $\Delta ER(\%)$ can be proved. The initial non diagonal elements ∂^{qw} of **K** are given as $\partial^{qw} = 0$ with $q \neq w$ and $\partial^{qw} = 1$ with $q = w$ (q,w = 1, 2, 3). One could argue that this assumption is unrealistic; it is, but it does not matter here. The relevant question is whether the width of a parameter distribution can be reduced by increasing T. Of course, if the level of ∂^{qw} also influences the width of parameter probability distribution, the level of $\Delta ER(\%)$ will also shift. But the level of $\Delta ER(\%)$ is not important here because for all three cases the same initial value for ∂^{qw} is given. For this reason, the influence of ∂^{qw} will be investigated later. The initially expected probabilities of default are drawn from the intervals $PD^{r=1} \in$ [0,1%; 0,3%], $PD^{r=2} \in$ [0,5%; 1,5%] and $PD^{r=3} \in$ [4,0%; 6,0%]. The initial inter rating class correlation coefficients are drawn from the intervals $\rho_{Probit}^{r=1,Asset} \in$ [0,17; 0,18], $\rho_{Probit}^{r=2,Asset} \in$ [0,15; 0,16] and $\rho_{Probit}^{r=3,Asset} \in$ [0,14; 0,15], and the sizes of the rating classes are drawn from the intervals $N^{r=1} \in$ [1.000; 1.200], $N^{r=2} \in$ [500; 700] and $N^{r=3} \in$ [1.100; 1.300].

Table 1 shows the average (Mean) and the standard deviation (SD) of $\Delta ER(\%)$ for each scenario and for each estimation method used. The reason for the size of $\Delta ER(\%)$ lies in the difference in size between VaR_α^o and VaR_α^{EU}. Hence the results for three diverse confidence levels α are also represented. As an example, let's have a look to the scenario T = 15 and $\alpha = 99{,}9\%$. If the ML- method is used, the amount of economic equity increase by 31.51% in average of the 50 simulated scenarios because of the estimation uncertainty. The standard deviation is 6.45% in this case.

Table 1:
Relationship between extra economic equity requirements caused by estimation uncertainty and the number of historical periods T

	T = 10			T = 15			T = 20		
α	95%	99%	99.9%	95%	99%	99.9%	95%	99%	99.9%
Mean (M1)	5.82%	19.60%	40.45%	3.94%	14.45%	29.10%	2.70%	11.02%	24.00%
SD (M1)	1.60%	2.29%	4.93%	1.18%	1.67%	4.77%	1.26%	1.33%	3.23%
Mean (M2)	7.63%	20.30%	40.01%	6.23%	15.46%	30.87%	4.76%	12.46%	24.91%
SD (M2)	1.15%	1.95%	5.04%	1.08%	1.99%	3.65%	1.27%	1.55%	3.65%
Mean (ML)	6.05%	20.20%	42.00%	4.70%	14.90%	31.51%	3.62%	12.59%	27.59%
SD (ML)	1.47%	2.81%	7.52%	1.47%	2.32%	6.45%	1.12%	2.33%	6.38%

20 See for example *Dannenberg* (2010); *Tarashev* (2009).

It is hardly surprising that the number of historical periods T which is used for parameter estimation has an important impact on $\Delta ER(\%)$. It is also clear that the selected confidence level α is very important for $\Delta ER(\%)$. This means that, up to a certain point of confidence level, estimation uncertainty can be eliminated by diversification. From this it follows that only the tails of the credit risk probability distribution are widened by estimation uncertainty. But the tails of risk distributions are particularly relevant in practice. Hence the relevance of estimation uncertainty increases with the confidence level. The standard deviation could be reduced by increasing the bootstrap sample size (x), but different manifestations of the initial parameters $PD^r, \rho_{Probit}^{r,Asset}, N^r$ could also explain a part of the shown standard deviation. The following section thus investigates whether a relationship between the level of initial parameters and estimation uncertainty exists.

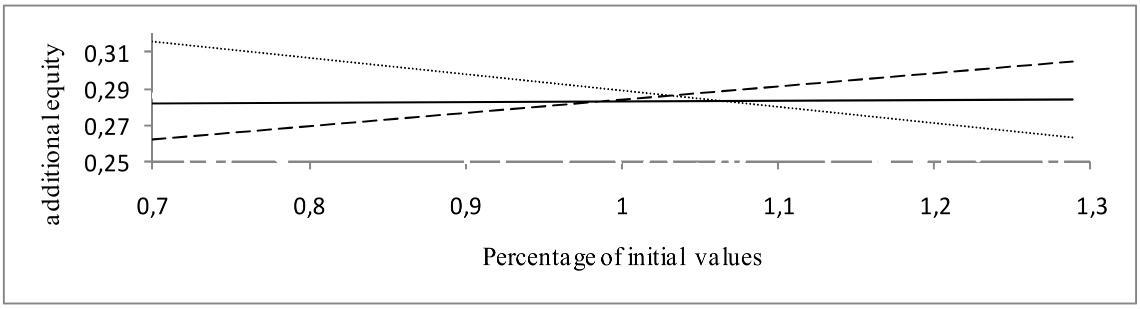
5.3 Relationship between Estimation Uncertainty and the Parameters $PD^r, \rho_{Probit}^{r,Asset}, N^r$

The question of whether differences in $\Delta ER(\%)$ in the samples could be caused by randomly drawn initial parameters $PD^r, \rho_{Probit}^{r,Asset}, N^r$ arises from the standard deviation measured in the previous section. In other words: are there relations between these parameters and estimation uncertainty? In order to answer this question here for each parameter, a simulation study is realized. The starting situation is defined as follows: $PD^{r=1} = 0{,}2\%$, $PD^{r=2} = 1{,}0\%$, $PD^{r=3} = 5{,}0\%$, $\rho_{Probit}^{r=1,Asset} = 0{,}16$, $\rho_{Probit}^{r=2,Asset} = 0{,}15$, $\rho_{Probit}^{r=3,Asset} = 0{,}14$ and $N^{r=1} = 1.100$, $N^{r=2} = 600$, $N^{r=3} = 1.200$. Depending on the research question, these starting parameters are varied. The number of historical periods is $T =$ 15. The elements of **K** are $\partial^{qw} = 0$ with $q \neq w$ and $\partial^{qw} = 1$ with $q = w$ (q,w = 1, 2, 3). The significance level of VaR is $\alpha = 99.9\%$.

In order to answer the question of whether relationships between a parameter and estimation uncertainty exist, with respect to the initial situation, each parameter is gradually increased in 50 steps from 70% to 130%. Hence, for each simulation study, 50 values of $\Delta ER(\%)$ are calculated. Regression lines are estimated to identify a relationship between parameters and $\Delta ER(\%)$. Thus, for each parameter $PD^r, \rho_{Probit}^{r,Asset}, N^r$ such a regression line can be determined. Figure 1 illustrates these regression lines using the method of moments M1. A similar result can be obtained using M2. The ML method reveals the same results in relation to PD^r and $\rho_{Probit}^{r,Asset}$. But in relation to rating class size N^r, the ML method shows a significantly negative relationship between this parameter and $\Delta ER(\%)$.

Clearly, a negative correlation between probability of default PDr and $\Delta ER(\%)$ exists. This means that estimation uncertainty is particularly important, especially in top rating categories. In percentage terms, therefore, more additional equity must be made available for the rating category AAA than for rating category BBB. The regression line slopes significantly negatively to the 1% level for all estimation methods used (M1, M2 and ML). The intra rating class correlations $\rho_{Probit}^{r,Asset}$ and $\Delta ER(\%)$ are positively correlated. The regression line slopes significantly to the 5% level (M1) respectively to the 1% level (M2, ML). This indicates that if the estimated value of the intra rating class correlation is large, the additional amount of equity will also be large.

Figure 1:
Regression lines for additional equity $\Delta ER(\%)$ using M1 for variation of PD^r (dotted line), $\rho_{Probit}^{r,Asset}$ (dashed line) and N^r (solid line)



As far as the size of the rating categories N^r are concerned, a significantly negative slope (1% level) can only be measured by using the ML method. Surprisingly, no effect is apparent when methods of moment are used. In the light of the attention given to this parameter in the literature on estimation uncertainty measurement using confidence intervals, this is an unexpected result. The reason for the insignificant relationship between N^r and $\rho_{Probit}^{r,Asset}$ could be the relatively large initial rating class sizes that are simulated here. Another reason could be an insufficient variation in size of rating classes during the simulation. Consequently, in an additional simulation a stronger variation of rating class size was chosen. This further simulation was accomplished where the rating class size of all rating categories was increased from $N^r = 100$ to $N^r = 900$ in 50 steps. For this simulation design a significantly negative slope (5% level) was observed. This suggests that a relationship between rating class size and $\Delta ER(\%)$ exists. But this relationship is relatively weak when methods of moment are used.

5.4 Relationship between Estimation Uncertainty and Grade of Inhomogeneity

In the previous section it was shown that a relationship between probability of default and the amount of additional capital exists, caused by estimation uncertainty. This section investigates whether increasing the difference between the probabilities of default of two rating classes also leads to an increase in $\Delta ER(\%)$. Such a result should be observed if the relationship between PD^r and $\Delta ER(\%)$ is nonlinear. For the purposes of this investigation, intra rating class correlation $\rho_{Probit}^{r,Asset}$ and rating class size are assumed to be identical in both classes $\rho_{Probit}^{r,Asset} = 0.15$ and $N^r = 1{,}100$ ($r = 1, 2$). The inter class correlation is assumed to be $\partial^{qw} = 0$ with $q \neq w$ and $\partial^{qw} = 1$ with $q = w$ ($q,w = 1, 2$). A rating history of length $T = 15$ is examined.

Figure 2:
Relevance of inhomogeneity to $\Delta ER(\%)$

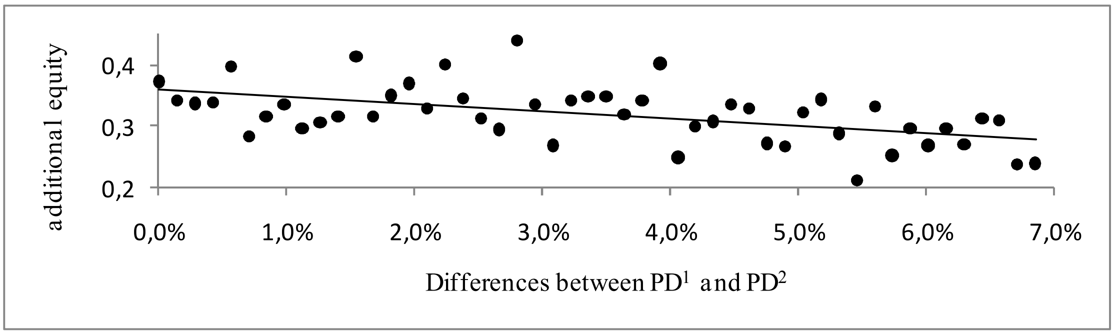


Figure 2 shows $\Delta ER(\%)$ for different grades of inhomogeneity using M1. The chosen confidence level of Value at Risk is $\alpha = 99.9\%$. All 50 analyzed portfolios have on average an expected probability of default of 4%, but the probabilities of default of the two rating classes are varied. For example, a difference of 2% in figure 2 means that $PD^1 = 3\%$ and $PD^2 = 5\%$. The slope of the regression line is significantly negative at the 1% level. This means that a bank with very different rating categories requires less additional equity than one with a more homogenous portfolio, even if the expected default rates of the portfolios of both banks are identical. Additional underlines the shown non-linear relationship between PD^r and $\Delta ER(\%)$ the relevance of estimation uncertainty for top-rating categories. Methods M2 and ML produce the same results.

5.5 Relationship between Estimation Uncertainty and Number of Rating Classes

In the previous sections, only portfolios with two or three rating classes were analyzed. This raises the following question: does the level of the simulated results also depend on the number of rating classes of a portfolio? This question is relevant because of a possible diversification effect. In other words, when considering a further rating class the additional equity amount will increase. But the additional effect of further rating classes in a portfolio $\Delta ER(\%)$ should decrease with the number of rating classes.

This assumption can be confirmed. The additional equity amount increases when a further rating class is added, but the slope of $\Delta ER(\%)$ decreases when the number of rating classes is increased. Table 2 indicates $\Delta ER(\%)$ for portfolios with different numbers of rating classes. In this investigation, inter rating class correlation is assumed to be $\partial^{qw} = 0$ with $q \neq w$ and $\partial^{qw} = 1$ with $q = w$ ($q,w = 1,\ldots,7$). The rating history is $T = 15$ and the number of rating classes varies between 1 and 7. By way of derivation from previous sections, the probability of default PD^r is here assumed to be identical in all rating categories ($r = 1,\ldots, 7$). This is in fact not an inhomogeneous portfolio but this assumption is useful when investigating the effect of the number of rating classes. Because of this assumption, it is unnecessary to investigate whether an observed result is caused by the chosen probability of default or not. The initial parameters for $PD^r, \rho_{Probit}^{r,Asset}, N^r$ are randomly drawn from uniform distributions with intervals $PD^r \in$ [0,1%; 6,0%], $\rho_{Probit}^{r,Asset} \in$ [0,14; 017] and $N^r \in$ [500; 1.300]. Because of considerable simulation efforts, only the methods M1 and M2 are used here.

Table 2:
Relationship between extra economic equity requirements caused by estimation uncertainty and the number of rating classes

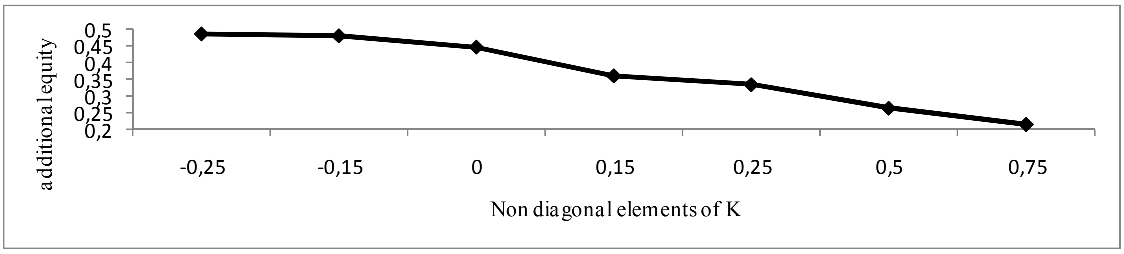
	α	R = 1	R = 2	R = 3	R = 4	R = 5	R = 6	R = 7
M1	95%	2.11%	4.90%	6.64%	7.89%	9.17%	10.16%	11.46%
	99%	12.89%	17.46%	19.78%	22.00%	24.07%	25.69%	27.45%
	99.9%	30.69%	36.95%	40.27%	43.24%	45.56%	47.40%	49.80%
M2	95%	2.10%	6.80%	10.96%	14.62%	17.64%	20.72%	23.46%
	99%	13.83%	20.28%	24.73%	29.17%	32.79%	36.11%	39.58%
	99.9%	32.48%	38.56%	43.67%	47.75%	51.66%	55.59%	58.96%

5.6 Relationship between Estimation Uncertainty and Inter Class Correlation ∂^{qw}

Up to this point, the non diagonal elements of **K** were assumed to be zero. This section now investigates whether **K** also has an influence on $\Delta ER(\%)$. A credit portfolio with five rating categories is considered. Seven different manifestations of ∂^{qw} are provided. For each given ∂^{qw} 50 random cases are simulated. Figure 3 shows the average of $\Delta ER(\%)$ of these 50 cases for each given ∂^{qw}. In each case, all non diagonal elements of **K** are assumed to be identical, in other words, $\partial^{12} = \partial^{13} = ... = \partial^{45}$. Figure 3 reflects the results using M2, but M1 produces the same findings. Owing to the simulation effort required, the ML method is not used. Again, the confidence level of Value at Risk is 99.9%. The initial parameters of PD^r and $\rho_{Probit}^{r,Asset}$ are randomly drawn from intervals $PD^{r=1} \in$ [0,1%; 0,3%], $PD^{r=2} \in$ [0,7%; 0,9%], $PD^{r=3} \in$ [1,0%; 2,0%], $PD^{r=4} \in$ [2,0%; 3,0%], $PD^{r=5} \in$ [3,0%; 5,0%] and $\rho_{Probit}^{r,Asset} \in$ [0,14; 0,15] with r = 1,…, 5. The rating class size is drawn from the interval $N^r \in$ [500; 1.200] with r = 1,…, 5. Figure 3 below indicates a negative relationship between inter rating class correlation and $\Delta ER(\%)$.

The reason for this finding could be an asymmetric parameter distribution of ∂^{qw}. Manifestations of ∂^{qw} can only be elements of the interval [-1; 1]. Therefore, if the estimated initial parameter value is positive, the parameter distribution is typically skewed to the left. If, however, the estimated initial parameter value is negative, the parameter distribution is skewed to the right in small credit portfolios. For example, if the estimated initial parameter value is ∂^{12} = 0.75, the interval that covers 99.9% of possible true values of ∂^{12} may be [-0.1; 0.95]. If the initial parameter value is ∂^{12} = -0.25, this interval may be [-0.75; 0.35]. In the first case the difference between the estimated parameter and the parameter which determines the Value at Risk (VaR_{α}^{EU}) may be 0.13 or 0.14. In the second case this difference may be 0.54 or 0.55. The larger the value of the inter rating class correlation, the more likely the common default of many borrowers becomes. Thus, the larger the difference between the initial value ∂^{qw} and the manifestation of the inter class correlation that determined VaR_{α}^{EU}, the larger the $\Delta ER(\%)$. Therefore, $\Delta ER(\%)$ decreases when ∂^{qw} increases.

Figure 3:
Relationship between inter class correlation ∂^{qw} and $\Delta ER(\%)$



6 Estimation Uncertainty in an Example based on Moody's Data

Finally, to illustrate the relevance of estimation uncertainty, an example based on the default rates of Moody's rating categories Baa, Ba, B and C is discussed. Although Moody's rating history covers more than 20 years,[21] in this example only credit defaults between 1989 and 2008 were considered. This limitation was imposed because a longer data history is unrealistic in the case of banks.

Moody's data provide the number of creditors and default rates for every year.[22] Based on these values, the number of defaults is calculated for every year. One must take into consideration the fact that this is only an approximate calculation as Moody's corrected the numbers of issuers by withdraw ratings. Based on the calculated number of defaults and the given number of issuers, the model is parameterized by the ML method. For the purposes of the risk analysis it is assumed that the size of rating classes in the forecast period corresponds to the size of rating classes in 2008. Notwithstanding the model assumptions in section 4, here the number of trials in Step (S6) is increased to x = 10,000 and in step (S12) to y = 1,000,000. These two credit risk distributions (considering and ignoring estimation uncertainty) do not differ in appearance. Differences are evident only in the tails. Figure 4 shows the upper 2.5% quantile of three possible loss distribu-

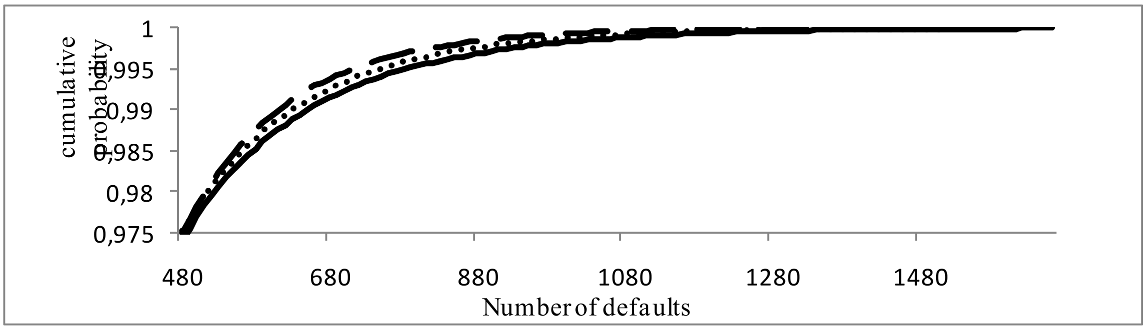
21 *Emery* et al. (2009), pp. 37 ff.

22 *Emery* et al. (2009), pp. 37 ff.

tions (dashed line: ignoring estimation uncertainty, solid line: considering estimation uncertainty using M2 and dotted line: considering estimation uncertainty using ML).

The expected number of defaults is 143. The Value at Risks $VAR_{99.9\%}$ are 1124 (M2), 1054 (ML), and 926 respectively (ignoring estimation uncertainty). It is assumed that the expected return on equity (τ) is 10%. Using equation (6) the equity requirement *ER* can be calculated. The equity requirement increases by 25.3% (M2) and 16.3% (ML) when estimation uncertainty is considered. On average, over all the rating classes, the interest rate increases by 56 (M2) and 36 (ML) basis points. If the earning target of equity is $\tau = 25\%$, the average interest rate would increase by 123 (M2) and 80 (ML) basis points. It is also noted that in particular rating categories the increase in interest rate may differ from the average increase shown here.

Figure 4:
97.5% quantile of cumulative distribution function of credit risk (dashed line: ignoring estimation uncertainty, solid line: considering estimation uncertainty using M2 and dotted line: considering estimation uncertainty using ML)



In addition to the costs of lending, estimation uncertainty is also important for instruments of financial markets such as Collateralized Debt Obligations (CDO). For instance, if a banker, who ignores estimation uncertainty, asserts that the probability of losses is smaller than 0.01% in the senior tranche of such an instrument, this would significantly underestimate the real loss probability of 0.06% (M2) and 0.04% (ML). Given the fact that further model risks exist, statements about the probability of losses of such instruments at such high significance levels should be regarded with great caution. Consideration of model risks could lead to a dramatic downgrade of AAA rated senior tranches.

7 Conclusion

In this paper, based on simulation studies of a multivariate Bernoulli mixture model, the relevance of estimation uncertainty was investigated. The research question was, 'which factors are related to the amount of required additional equity caused by estimation uncertainty?'. In addition to the relationships between estimation uncertainty and number of historical periods and size of rating categories already discussed in the literature, relationships between $\Delta ER(\%)$ and probability of default, inter and intra rating class correlation, number of rating categories and also grade of inhomogeneity are shown here. The skewness of parameter distributions is presumed to have accounted for these findings. From this it follows that symmetrical distributions, like normal distributions in real credit portfolios, are inappropriate when considering estimation uncertainty if a high confidence level of Value at Risk is chosen.

Presumably, model risks will gain relevance in banking regulation. This raises the question of which institutes are extraordinarily affected by model risks. Based on the investigations presented here, it is advised that, in particular, banks with credit portfolios dominated by debtors with good or very good ratings require a proportionally larger amount of extra equity to cover model risks. A further point of interest which this investigation raises is that estimation uncertainty is more relevant to banks whose fundamental credit risk is low (debtors with good ratings, low intra class correlations) than to those with high credit risks.

Compared to the simulation results in section 5, for the example based on Moody's data with 16% - 25%, a relatively low value for $\Delta ER(\%)$, is calculated in section 6. The reasons for these findings are the high probabilities of default of the rating categories B and C. Since these rating categories dominate the risk assessment they also dominate the assessment of estimation uncertainty; the calculated $\Delta ER(\%)$ primarily depends then on the effects of estimation uncertainty on these rating categories. On the other hand, the portfolio is relatively inhomogeneous which also limits the relevance of estimation uncertainty. In addition, high inter class correlations lead to a reduction in $\Delta ER(\%)$.

The model risk estimation uncertainty can be reduced by diversification. In this paper, this effect was shown by increasing the number of rating categories. Furthermore, it can be assumed that risk drivers of estimation uncertainty are relatively uncorrelated. Hence, if estimation uncertainty of one rating category is caused by low probability of default, and of another rating category by size of rating class, presumably these risk drivers are uncorrelated and therefore a diversification effect can be assumed. Therefore, if different risk drivers determine the model risk, a diversification effect should exist.

In this study, three different estimation methods were used. It was found that $\Delta ER(\%)$ also depends on the chosen estimation method. That means that estimation uncertainty is only one source of model risk. It was not the focus of this paper to investigate whether the model or the estimation methods used are the best choices to assess the risk of credit portfolios, but it is evident that an additional amount of equity is necessary for other model risks. The significance of a calculated Value at Risk should thus be interpreted with caution, particularly if the confidence level is above 99%.

Appendix 1

Three methods are used here to estimate the model parameters. First, the parameters of the marginal totals $F_{\tilde{\pi}_t^r}\left(\pi_t^r\right)$ are estimated using the canonical maximum likelihood method[23] (ML method). Based on the results of the first step, the Matrix **K** can be estimated. The maximum likelihood function used in the first step to estimate the parameters of the marginal totals is derived from formula (4):[24]

$$
\begin{aligned}
& l\left(PD^r, \rho_{Probit}^{r,Asset}, N_{t=1}^r, \ldots, N_{t=T}^r, H_{t=1}^r, \ldots, H_{t=T}^r\right) = \\
& \qquad \sum_{t=1}^{T} \ln\left(\int_0^1 \exp\left(H_t^r \cdot \ln\left[\Phi\left(\frac{\Phi^{-1}\left(PD^r\right) - \sqrt{\rho_{Probit}^{r,Asset}}\,\Phi^{-1}\left(\upsilon\right)}{\sqrt{1-\rho_{Probit}^{r,Asset}}} \right) \right] \right.\right. \\
& \qquad \left.\left. + \left(N_t^r - H_t^r\right) \cdot \ln\left[1 - \Phi\left(\frac{\Phi^{-1}\left(PD^r\right) - \sqrt{\rho_{Probit}^{r,Asset}}\,\Phi^{-1}\left(\upsilon\right)}{\sqrt{1-\rho_{Probit}^{r,Asset}}} \right) \right] \right) d\upsilon \right).
\end{aligned}
\tag{A1}
$$

This function is maximized numerically for each rating category. The maximization is based on information about historical defaults H_t^r and size of rating categories N_t^r at time t (t = 1, …, T). In this way, T historical periods are available for estimation. The function is maximized over PD^r and $\rho_{Probit}^{r,Asset}$. The estimators of the parameter are symbolized by $\widetilde{PD}_T^r$ and $\tilde{\rho}_{Probit}^{r,Asset}$.

In this instance, dependencies between rating classes are modeled by Gaussian Copula with correlation matrix **K**. The maximizing function is:[25]

$$
\begin{aligned}
& l\left(PD^{r=1}, \ldots, PD^{r=R}, \rho_{Probit}^{r=1,Asset}, \ldots, \rho_{Probit}^{r=R,Asset}, \mathbf{K}\right) = \\
& \quad \sum_{t=1}^{T} \ln\left(\mathbf{\Phi}_{\mathbf{K}}\left(\Phi^{-1}\left(F_{\tilde{\pi}_t^1}\left(\pi_t^1\right)\right), \Phi^{-1}\left(F_{\tilde{\pi}_t^2}\left(\pi_t^2\right)\right), \ldots, \Phi^{-1}\left(F_{\tilde{\pi}_t^R}\left(\pi_t^R\right)\right) \right) \right),
\end{aligned}
\tag{A2}
$$

with

$$
\mathbf{\Phi}_{\mathbf{K}}\left(\Phi^{-1}\left(F_{\tilde{\pi}_t^1}\left(\pi_t^1\right)\right), \ldots, \Phi^{-1}\left(F_{\tilde{\pi}_t^R}\left(\pi_t^R\right)\right) \right) = \frac{1}{\sqrt{\det\left(\mathbf{K}\right)}} \cdot \exp\left(-\frac{1}{2} \xi_t{}'\left(\mathbf{K}^{-1} - \mathbf{I}\right) \xi_t \right) \tag{A3}
$$

and with $\xi_t = \left(F_{\tilde{\pi}_t^1}\left(\pi_t^1\right), F_{\tilde{\pi}_t^2}\left(\pi_t^2\right), \ldots, F_{\tilde{\pi}_t^R}\left(\pi_t^R\right)\right)'$. The letter **I** denotes the identity matrix. By replacing the random probabilities of default π_t^r with historical default rates DR_t^r

23 *Cherubini/Luciano/Vecchiato* (2004), p. 160.

24 *Frey/McNeil* (2003), p. 81.

25 *Cherubini/Luciano/Vecchiato* (2004), p. 155 and p. 160.

and inserting the estimated parameters of the marginal totals, one can estimate the correlation matrix **K**. The default rate is calculated by:

$$DR_t^r = \frac{H_t^r}{N_t^r}, \text{ with } r = 1, \ldots, R \text{ und } t = 1, \ldots, T. \tag{A4}$$

The matrix of Default Rate is:

$$\mathbf{DR} = \begin{pmatrix} DR_1^1 & \ldots & DR_1^R \\ \vdots & & \vdots \\ DR_T^1 & \ldots & DR_T^R \end{pmatrix}. \tag{A5}$$

One of the methods of moment (M1) used here is described by *Höse* (2007).[26] This method assumes time invariant rating class sizes $N_t^r = N^r$. Although this assumption is not entirely realistic, this method is used here because of its rapid computability. The expected probability of default of a rating category PD_r is estimated by:

$$\widetilde{PD}_T^r = \frac{\mathbf{1DR}^r}{T}, \tag{A6}$$

where **1** denominates a 1×T row vector where all elements are one and $\mathbf{DR^r}$ denominates the vector of historical default rates of rating category r. The variance-covariance matrix of probability of default is estimated by:

$$\hat{\mathbf{\Sigma}}_T = \tilde{\sigma}_T^{qw} = \frac{\mathbf{DR'DR} - \dfrac{\mathbf{DR'1'1DR}}{T}}{T-1} \text{ with } q,w = 1, 2, \ldots, R. \tag{A7}$$

The asset-correlation $\rho_{Probit}^{r,Asset}$ of a rating category can be calculated by the numerical solution of the equation:

$$\Phi^{Biv}_{\tilde{\rho}_{Probit,T}^{r,Asset}}\left(\Phi^{-1}\left(\widetilde{PD}_T^r\right), \Phi^{-1}\left(\widetilde{PD}_T^r\right)\right) = \tilde{\sigma}_T^{rr} + \left(\widetilde{PD}_T^r\right)^2. \tag{A8}$$

where $\Phi^{Biv}_{\tilde{\rho}_{Probit,T}^{r,Asset}}$ denotes a bivariate normal distribution with correlation coefficient $\tilde{\rho}_{Probit,T}^{r,Asset}$. The elements of **K** (∂^{qw}) can be determined by numerically solving the equation:

$$\int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}\left(\widetilde{PD}_T^q\right) + \sqrt{\tilde{\rho}_{Probit,T}^{q,Asset}}\, z}{\sqrt{1-\tilde{\rho}_{Probit,T}^{q,Asset}}}\right) \cdot \Phi\left(\frac{\Phi^{-1}\left(\widetilde{PD}_T^w\right) + \sqrt{\tilde{\rho}_{Probit,T}^{w,Asset}} \cdot \tilde{\partial}_T^{qw} z}{\sqrt{1-\tilde{\rho}_{Probit,T}^{w,Asset} \cdot \left(\tilde{\partial}_T^{qw}\right)^2}}\right) \cdot \phi(z)\, dz = \tilde{\sigma}_T^{qw} + \widetilde{PD}_T^q \widetilde{PD}_T^w \tag{A9}$$

26 *Höse* (2007), pp. 89 ff and pp. 180 ff.

with $q,w = 1, 2, \ldots, R$ and $q \neq w$. The density function of Gaussian distribution is represented by $\phi(z)$.

As an alternative to method M1, a second method of moments (M2) is used here. Estimation of probability of default corresponds to the method described in equation (A6). The asset-correlation is estimated by numerically solving the equation:[27]

$$\Phi^{Biv}_{\rho^{r,Asset}_{Probit,T}}\left(\Phi^{-1}\left(PD^r\right),\Phi^{-1}\left(PD^r\right)\right)=PD^r_{i,j}, \tag{A10}$$

where the joint probability of default $PD^r_{i,j}$ of two borrowers of a rating category is estimated by:

$$\widetilde{PD}^r_{i,j}=\left(\widetilde{PD}^r_T\right)^2+\frac{\frac{1}{T-1}\sum_{t=1}^{T}\left(DR^r_t-\widetilde{PD}^r_T\right)^2-\frac{\widetilde{PD}^r_T\left(1-\widetilde{PD}^r_T\right)}{T}\sum_{t=1}^{T}\frac{1}{N^r_t}}{1-\frac{1}{T}\sum_{t=1}^{T}\frac{1}{N^r_t}}. \tag{A11}$$

Equation (A2) could be used in the estimation of **K**. Since the ML method is very time consuming, the correlation matrix **K** is estimated in method M2 by the Spearman's rank correlation coefficients.[28]

27 *Gordy* (2000), p. 146; *Frey, McNeil* (2003), p. 78; *McNeil, Frey, Embrechts* (2005), p. 376.

28 *Cherubini, Luciano, Vecchiato* (2004), pp. 95 ff.

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