The Valuation of Corporate Debt with Default Risk

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Abstract. This article values equity and corporate debt by taking into account the fact that in practice the default point differs from the liquidation point and that it might be in the creditors' interest to delay liquidation. The article develops a continuous time asset pricing model of debt restructuring which explicitly considers the inalienability of human capital. The study finds that even though in general the creditors will not liquidate the firm on the incidence of default, but nevertheless would liquidate the firm prematurely relative to the first best threshold. This agency problem leads to the breakdown of the capital structure irrelevance result.

1. Theoretical Foundation

The literature on the pricing of defaultable bonds started with Merton (1974) who applied the contingent claims valuation insight of Black and Scholes (1973) to the pricing of corporate bonds. He obtained closed form valuation expressions for (zero-coupon) risky bonds, by taking the lower reorganization boundary as given. Thus in his model, default occurred at maturity if the value of the firm was less than the payment promised to the bondholders. Furthermore, at maturity the compensation received by the creditors is fixed and they receive the minimum of the value of the firm or the contracted payment. In the event of a default, the control of the firm is transferred to the creditors and this default point can also be interpreted as a liquidation point with zero bankruptcy costs. Thus, in Merton's model, both the lower reorganization boundary and the compensation received by the bondholders is taken to be exogenous. In practice, however, the compensation received by the creditors may vary continuously with the value of the firm if the firm is in financial distress. Furthermore, Merton does not make the distinction between default and liquidation.

There have been a number of extensions of the Merton model. Black and Cox (1976) incorporate bond indenture provisions. Jones, Mason and Rosenfeld (1984) consider multiple issue of callable coupon debt. In both of these models however the induced lower boundary at which the firm is liquidated is taken to be exogenous. Further, default is again tantamount to bankruptcy in these models.

Leland (1993) and Leland and Toft (1996) extend Merton's analysis by endogenising the lower reorganization boundary. By assuming that the equityholders will always issue equity to prevent a default, they obtain expressions for the values of risky debt via the smooth-pasting condition. Thus in their models, the equityholders keep on issuing equity (when necessary) to avoid a default until the value of equity falls to zero. Thus default occurs when the value of equity falls to zero and this default point is again synonymous

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with bankruptcy. They do admit that default need not lead to bankruptcy in practice, but nevertheless they interpret their default point as the bankruptcy point. Since their default and bankruptcy points are in effect the same, the compensation received by the creditors in the event of a default is again exogenous and is just equal to value of the firm adjusted for bankruptcy costs.

Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) also consider endogenous bankruptcy and they model the strategic behaviour of debtors. In their models, the debtors act strategically and always try to pay as low a coupon as possible. In good times when the liquidation value of the firm is high, the debtors will not pay lower than the contracted amount as they would realise that it would then be in the creditors' interest to reject their offer and liquidate the firm. However, the debtors might underperform the debt contract even if the firm is not experiencing any liquidity problems. They will do this when the liquidation value of the firm is not sufficiently high and thus when subsequently it would be not in the creditors' interest to reject the offer. Thus in their models, the debtors might default continuously and they will continue to do so until the creditors finally reject the offer. At this point the firm will be liquidated.

Thus their models are similar to our model in the sense that endogenous bankruptcy occurs and that this bankruptcy point will in general be different from the default point. However there are a number of differences. We do not consider strategic debt service. In our model the debtors do not act strategically because once a default occurs, debt covenants are triggered off and this gives the creditors an option to either liquidate the firm or keep the firm running. In either case the debtors then only get the value of their outside option. Thus as long as the firm is in default the debtors' payoff is limited to their outside option and the creditors get most of the bargaining power as they can always threaten to liquidate. However the creditors are aware of the fact that they cannot run the firm without the manager because of her inalienable human capital. We therefore explicitly model the manager's inalienable human capital. Subsequently, if the creditors decide to keep the firm alive, they have to ensure that the manager stays in the firm. They can do this only if they offer the manager at least the value of her outside option. Hence in the renegotiation game of our model, the creditors have bargaining power because they hold an option to liquidate once the firm enters a default while the debtors' bargaining power stems from their inalienable human capital.

One important difference between Anderson and Sundaresan (1996) and our model is that unlike their model, the creditors in our model act strategically once a default occurs. Thus if they decide not to liquidate then they will just pay enough to the manager so as to retain her human capital and will thus in effect become the residual claimants until the firm exits default.

Mella-Barral (1999) obtains some results which are quite similar to ours. Like our model, one of the objectives of his model is to explain why default rarely coincides with liquidation. Like us, he also shows why it might be rational for the creditors not to liquidate even if the debtors do not have any bargaining power. This is because renegotiating the debt can actually increase the market value of debt by avoiding ill-timed liquidation.

However, there are a number of important differences. Mella-Barral considers two mirror games where either the creditors are in a position to make take-it-or leave-it offers to debtors or the debtors can have the first mover advantage. In the former case the debtors do not have any bargaining power and once a default occurs, the creditors themselves make self imposed concessions. The concessions are just enough to induce the debtors not to default if the firm was being liquidated at an inefficiently early stage.[1] Thus in his model the debtors do not have any bargaining power if the creditors have the first mover advantage. On the other hand in our model, the debtors always have bargaining power over the creditors as they have perfectly inalienable human capital. They can always threaten (the creditors) to leave the firm if they do not get at least the value of their outside option. Furthermore, in our case liquidity problems determine the default point. Conversely, Mella-Barral assumes that liquidity problems do not have any influence on the default point as the debtors can keep on issuing equity to avoid a default. Subsequently in his model, default is endogenous and is the point where it is optimal for the debtors to irreversibly exchange their current claim for a residual claim which they will get on bankruptcy. In our set-up default occurs because of liquidity problems and is therefore exogenous. Further, in our case there is no irreversible exchange of claims and either party may get their original claims if the firm manages to exit default. In Mella-Barral's analysis, a capital structure irrelevance result holds after renegotiation. However, the Modigliani Miller Theorem will in general not hold in our setting.

Perhaps the most significant difference between the two analyses is that in Mella-Barral's model once a default occurs it is necessary to offer new debt contracts to investors in order to exit the default state. The alterations in the debt contract are not temporary. If a default problem arises again, then another debt contract has to be drawn up. Thus Mella-Barral allows for an unlimited sequence of new contractual arrangements. Thus in his analysis new debt contracts have to be drawn up continuously every time the firm enters default. This is not the case in our framework. In our model, if the firm goes into financial distress and thus defaults, then the firm undergoes a reorganization period where a trustee or an administrator tries to solve the firm's financial problems. If they manage to bring the firm out of default then the old contract governs again. If however, the firm's financial position worsens then finally the firm will be liquidated. Hence any alterations in the payoff functions of both the debtors and the creditors on default are only temporary and are thus reversible. This is consistent with the US Trust Indenture Act of 1939 which prohibits firms from permanently changing the 'core' terms of the bond indenture, which include the principal amount, the interest rate and the stated maturity, unless all the creditors agree unanimously.[2]

Recently a number of other studies have also incorporated endogenous liquidation. Leland (1998) extends his earlier work by considering the effects of asset substitution on debt pricing. As in his earlier work he does not make the separation between default and liquidation. Hege and Mella-Barral (2000) extend Mella-Barral (1999) by taking into account multiple creditors. Christensen et al. (2001) consider callable perpetual debt where both the upper and lower reorganization points are derived endogenously. Their work is an extension of the strategic debt service models of Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) to a set-up where the entire debt rather than just the current coupon payment is renegotiated. This feature is similar to our model, as in our framework all the coupon payments, rather than just the current one, are altered in the default region. However, unlike Christensen et al. this alteration is not permanent and is reversible.

[1] Mella-Barral (1999) makes a distinction between 'early' liquidation and 'late' liquidation. In his model 'early' liquidation occurs if the debtors decide to default too early relative to the first best liquidation point. 'Late' default occurs if the debtors do not default even if it is (first best) efficient to do so.

[2] If the firm goes to Chapter 11, then these core terms can be altered if a two-thirds majority by value and a simple majority by number is reached within each class of creditors. However unanimity is required outside of Chapter 11.

With respect to the debt pricing literature our main contribution is to provide a clear separation between default and liquidation which is consistent with existing bankruptcy regimes. In much of the current debt pricing literature, the entrepreneur maximising the value of equity makes the liquidation decision. However the separation of default and liquidation in our model enables us to delegate the liquidation decision to the creditors. Thus in our model default is triggered by the manager, whilst the liquidation decision is made by the creditors. This seems more natural as the occurrence of default triggers debt covenants, which in turn might lead to liquidation by creditors.

We construct a model where default in general would not lead to immediate liquidation. In our model, once a default occurs the debtors lose most of their control rights and the creditors in effect become the residual claimants as long as the firm remains in financial distress. On the occurrence of a default the control rights are passed to a creditors' representative who then decides whether to continue or to liquidate. This decision will be in the best interest of the creditors. The firm will be liquidated by the representative creditor only if from the creditors' point of view there is no benefit in keeping the firm alive. This will be the case if the firm's financial position is very dismal or if the firm's insolvency further aggravates after default. However, as long as the firm is not liquidated the debtors preserve control of the management of the firm because of their inalienable human capital. If the firm manages to exit default then the old contract governs again and the debtors then retain all of their control rights and are once again the residual claimants.

This feature of our model, whereby the creditors on default appoint a representative to make optimal decisions on their behalf but where the debtors still make the day to day management decisions of the firm is a good approximation to most of the existing bankruptcy regimes. For instance in the US under Chapter 11, the debtors maintain their control of the management of the firm as the debtor-in-possession or are at least aided by a trustee appointed by the court. Thus even after default the value of equity in practice is not zero. This is consistent with our model where the value of equity falls to zero only when the firm is finally liquidated. However once the firm files under Chapter 11 the management is subject to detailed supervision by the court.

In the UK most banks hold a floating charge over the assets of a company and if the firm is unable to meet its obligations then the banks have the power to appoint an administrative receiver who then supervises the running of the firm and who has the power to put the company into liquidation. Alternatively, the creditors in UK can resort to formal reorganization by going to court. The court then appoints an administrator who has all the powers vested in the board of directors.[3] Administration is succeeded by liquidation only if the company is unable to survive as a solvent going concern. The UK Bankruptcy Code closely resembles the South African "judicial management" and the Australian "official management" and gives relatively more powers to the creditors vis-a-vis the US Chapter 11.[4]

In France, the judicial arrangement (Redressment Judiciare) consists of two stages; the observation stage and the execution stage. The observation stage in terms of our model can be interpreted as the reorganization stage whereby both parties try to reach an agreement to keep the firm alive. The execution stage is the liquidation stage and

[3] Section 17 of the UK Insolvency Act 1986 provides that the administrator shall on his appointment take control of all the property to which the company appears to be entitled until the firm is either restored to good health or is liquidated.

[4] However after the 1994 reform, Chapter 11 returned some powers to the creditors.

it occurs when reorganization fails. When a firm defaults and goes to court, then the court appoints an administrator, a creditor's representative and a supervisory judge. The control rights pass from the debtors to the administrator working with the judge. However the management retains control of the running of the firm. In Germany again as long as the firm is not liquidated, the debtors retain control of management but are supervised by an administrator representing the creditors. Thus the structure of our model closely resembles most of the existing bankruptcy regimes.

Like Hart and Moore (1994) a central feature of our model is the inalienability of human capital. The bargaining power of the manager in our model comes from her ability to threaten the creditors to repudiate the contract by withdrawing her human capital. However one of the assumptions underlying Hart and Moore's theory of debt is that the manager does not have any outside options and hence has a zero outside wage. On the contrary, we do not make that assumption and we assume that the manager always has a valuable outside assumption. This is quite realistic and as argued by Rajan and Zingales (2000) with the increase in competition physical assets have become less unique and managers today have many outside options.

Rajan and Zingales (2000) in their article, "The Governance of the New Enterprise" argue that powerful forces are changing the nature of the firm and the increased importance of human capital has led to the breakdown of the traditional vertical integrated firm. Rajan and Zingales (2000) reflecting on the changing nature of the modern enterprise argue:

> ...perhaps the most significant change has been to human capital. Recent changes in the nature of the organisations, the extent and requirements of markets, and the availability of financing have made specialised human capital much more important, and also much more mobile. But human capital is inalienable, and power over it has to be obtained through mechanisms other than ownership.

Our debt pricing model also has interesting implications for corporate finance as discussed in Section 3 of the paper. With respect to the corporate finance literature our main contribution is to show that the Irrelevance Theorem will not hold even in the absence of informational asymmetries, coordination problems, financial distress costs and taxes. We identify the first best liquidation point and show that the introduction of debt in the capital structure is conducive to an agency problem whereby creditors liquidate the firm prematurely.

The rest of the paper is organised as follows. Section 2 constructs a continuous time pricing model of the levered firm. We model the reorganization process and obtain analytical solutions for equity and corporate debt in terms of the endogenous bankruptcy point. We show that the default trigger will in general be different from the point of liquidation. In Section 3 we value the unlevered firm in the same set-up and show that in general capital structure irrelevance will not be obtained. We then characterize the first best liquidation point and offer an explanation as to why the Irrelevance Theorem breaks down. Section 4 provides a discussion of our results. Finally, Section 5 concludes.

2. The Model

2.1. The Basic Set-up. We model a firm which is run by one owner-manager endowed with specialised human capital. This human capital can be interpreted as a technical skill which is valuable to the firm and without which the firm cannot be run.

Alternatively the owner-manager can be thought of as possessing an essential asset which is vital for the operations of the firm. The manager has many outside options and her reservation income is a. The value of her outside option, a, can therefore be considered as her income if she decides to move back to her old job. Further, the manager requires a minimum wage of at least a and she will leave the firm at any period if her income is less than the sustenance level a. Hence the manager will consume at least a per period but will consume more if the cash flow of the firm is sufficiently high given that the owner will have residual claim in these good states of the world.

The owner has limited wealth and therefore requires financing for the firm's project. This financing is arranged through the issue of a debt contract.[5] We assume that the owner issues perpetual debt to finance the project. The terms of the debt contract require that a coupon payment, b, be paid to creditors per instant of time. We do not allow debt service to be funded through the issue of new securities or through asset sales. To abstract from any coordination problems we assume that there exists a representative creditor who makes optimal decisions on behalf of a cohesive group of creditors.[6]

We suppose that capital markets are frictionless and that there are no informational asymmetries between agents. Further, all agents are risk neutral and the term structure of interest rates is flat with a nonstochastic rate, r. The assumption of risk neutrality is without much loss of generality. The model can be developed using risk neutrality with ordinary probabilities, or alternatively under risk aversion with risk neutral probabilities. (See Harrison and Kreps (1979) for a discussion.) Finally, we assume that the liquidation value of the firm's project is given by K.

2.2. The critical default trigger. The firm's underlying state variable is cash flows, x, which follow a geometric brownian motion, i.e.

$$dx = \alpha x dt + \sigma x dw \tag{1}$$

where the parameters α and σ represent the drift and volatility terms respectively and dw is the increment of a standard Wiener process. Note that we do not assume that there exist traded securities such that any new claim can be priced by dynamically replicating already existing securities. Hence we do not require the completeness of markets and the stochastic changes in x need not be spanned by existing assets in the economy. Many corporate finance valuation models use the classical contingent claims pricing approach and use the value of the unlevered firm as the basic state variable. However contingent claims analysis assumes that markets are sufficiently complete and as pointed out by Christensen et al. (2001) it is impossible to have both the unlevered firm and the optimally levered firm to coexist as trading assets.

In our model, default occurs because of liquidity problems. This is because like Anderson and Sundaresan (1996) we assume that debt service is met out of cash flows and that the firm cannot issue additional equity or debt to avoid a default. This is not a very stringent assumption as it might first appear. In practice, debt covenants frequently restrict the issue of additional debt with senior or equal status. Similarly, loan indentures quite

[5] The debt contract is thus justifiable given the limited wealth of the owner.

[6] This is a reasonable assumption which is consistent with most of the exsiting bankruptcy codes. For instance, Chapter 11 provides an automatic stay against creditors' claims to avoid credior harrassment during the reorganization process. Thus creditors are prevented from foreclosing on their collateral. Nevertheless coordination problems can exist during the voting process (unless a 'cram down' is imposed) or in out-of-court restructurings.

often forbid the liquidation of firm's assets by owners as this could potentially undermine collateral values.

To simplify our analysis we also do not allow the firm to issue junior debt or equity. In some respects this brings us closer to reality. It is very rare for firms in financial distress to issue equity so as to avoid a default on its financial obligations. In a rational market, agents would realise that equity is being issued to prevent a default and thus any sale of equity would further decrease the suppressed value of shares.[7] Firms can in principle issue junior debt to avoid a default but this would be very costly for the issuing firm. The reason is that if a firm is in financial distress then the addition of further debt in its financial structure would just increase the debt servicing obligations of the firm and hence increase the likelihood of a default in future periods. Further, as argued by Anderson and Sundaresan (1996) the fixed costs relating to the issue of securities are likely to be higher for firms in financial distress and hence it may not be feasible for the firm to issue new securities. For all these reasons the addition of new claimants would most likely just delay default for a short period of time rather than prevent it. Nevertheless extending the model to allow for the issue of new securities would be an interesting area for further research.

Since in our model, default occurs because of liquidity problems facing the firm, hence the critical default point is determined exogenously. Default occurs whenever the firms current cash flows are not enough to meet its debt obligations. We conjecture the following.

Conjecture 1. Assume that the firm has issued perpetual debt with coupon payment b and principle b/r.[8] Suppose that the value of the manager's outside option is given by a. Let $\hat{x}$ denote the critical level of cash flows below which the firm will default on its debt payments and suppose that the firm cannot issue new securities or liquidate assets to service its debt. Then the critical default point of the firm is given by

$$\hat{x} = a + b \tag{2}$$

It is quite straightforward to see why Conjecture 1 holds if we suppose for the moment that the debtors will not act strategically when servicing their debt payments. Clearly if debtors do not act strategically then the default point will be given by Eqn. (2). This is because the owner needs to consume at least a per period. Thus in every period the owner will first take out a portion a of the cash flows before servicing its debt. Since the owner is obliged to pay a coupon of b period to the bondholders, thus default will occur whenever the cash flow falls below $a + b$. Thus the critical default point is $\hat{x}$.

Once a default occurs, debt covenants are triggered which gives creditors the right to liquidate. However the creditors might not find it in their interest to liquidate the firm and hence might want the firm to continue. Hence the occurrence of a default gives the creditors an option to liquidate which the creditors may or may not exercise.

A key feature of our model is that if the creditors decide to continue then the control rights are essentially transferred to the creditors as long as the firm is in the default state. For our purposes, control rights can be defined as the right to control the distribution

[7] This would also be true in the presence of asymmetric information in financial markets. Myers and Majluf (1984) argue that the issue of equity is taken as a bad signal by market participants and this therefore depresses stock prices.

[8] Like Mella-Barral and Perraudin (1997) we assume that the debt principal is b/r. This is purely for expositional purposes and is without much loss of generality.

of the firm's resources.[9] The reason why a transfer of control rights occurs on default is that as long as the firm is in default the creditors are getting less than their contracted payments and hence have the first right to the firms cash flows.[10] We claim the following.

Claim 1. The debtors will not default unless faced by liquidity problems and hence will not act strategically if a transfer of control rights occurs from the debtors to the creditors once the firm enters default. Hence, Conjecture 1 is true.

To show why the owner will not act strategically, suppose on the contrary that the owner decides to default even if the firm is in a healthy state and hence can meet its debt obligations. Suppose the owner decides to underperform the contract and pays less than the contracted coupon payment, b, to the creditors and further threatens to withdraw her inalienable human capital if the creditors do not accept her offer. If the owner pays an amount slightly over rK to the creditors, then the creditors would not find it in their interest to liquidate as liquidation will just fetch them a scrap value of K.[11] However, the creditors can decide not to exercise their option to liquidate and would actually prefer to let the firm continue if its fundamentals are strong enough. If the creditors continue then they will realise that the owner's threat to withdraw her human capital is not credible, given that they would now have the control rights. Once the creditors get the control rights they will offer the owner an amount slightly over a such that the owner will then have to accept the creditors' offer. The owner will foresee this outcome ex ante and will realise that her threat to withdraw her human capital is not credible. Hence the subgame perfect strategy of the owner will be not to default unless faced by liquidity problems. Thus Conjecture 1 is consistent with a subgame perfect Nash equilibrium.

A key distinction between our model and the strategic debt servicing models of Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) is that in our case a transfer of control rights occurs from the debtors to the creditors as long as the firm remains in default. It is because of this shift of the control rights that the debtors in our set-up do not act strategically. They are aware of the fact that on default, they will lose their control rights and hence will then be limited to the value of their outside option. Hence the debtors do not find it advantageous to strategically service the debt. However if the debtors are always in charge of the control rights of the firm then they will find it worthwhile to act strategically as in that case they can afford to make a take-it-or-leave-it offer and the creditors will not be able to respond with an offer of their own.

2.3. The reorganization process. Once the cash flows fall below the critical level $\hat{x}$, a default occurs and debt covenants are triggered. The creditors then have an option to either let the firm continue or to stop and liquidate. We thus solve for the creditors' optimal stopping problem. Before solving for the creditors' optimal stopping rule we need to consider the game between the creditors and the manager on the occurrence of a default. We thus embed a game theoretic approach in our continuous time asset pricing model to ascertain the equilibrium payoffs in different states of the world. The outcome of these

[9] Strictly speaking a distinction should be made between control rights and cash flow rights. The latter would be the right over the firm's cash flows whilst the former would be the right to control the liquidation decision. This difference does not matter in our case as creditors have both control rights and cash flow rights when the firm is in default.

[10] This can be interpreted as the appointment of a creditors' respresentative to oversee management and to ensure the protection of creditors' interest.

[11] rK will be less than b if the debt is risky.

games critically depend on the relative bargaining power between the two parties. Unlike most of the debt pricing models which give all the bargaining power to either the creditors or the debtors, we consider a case where both parties have some bargaining power during the reorganization process.

As argued by Rajan and Zingales (2000), there are different sources of power. Power can be derived through the rules of the bargaining process. For instance, the party with the first mover advantage might exert some power on the other. Alternatively, a negotiator can exert power if she possesses some valuable resource which is required in the production process. These are the two sources of power that we consider in our game.

In the game that we consider, creditors act as the Stackelberg leader and make take-it-or-leave-it offers to managers. Once a default is triggered, creditors can either continue or liquidate the firm. If they decide to continue than they get the control rights. They then realise that if they pay anything less than a to the owner-manager then the manager will take her human capital and leave. Further they are aware that the manager's human capital is inalienable and the firm cannot be run without her specialised knowledge. Hence the creditors pay the manager just enough so as to retain her human capital. They do this by offering the manager the value of her outside option. The manager hence accepts the offer. Thus rejecting never occurs in equilibrium. Hence in the subgame perfect equilibrium, the creditors offer an amount a in every period to the manager which the manager accepts. This will be the case as long as the firm is in default.

Note that the creditors derive their bargaining power from their option to liquidate the firm. It is this option which gives them the first mover advantage. The manager on the other hand influences the bargaining process because of her inalienable human capital. The higher the value of her outside option the more will be her share of the bargaining power. However if the value of her outside option is too high, then the creditors might be better off by just liquidating the firm.

Once a default occurs, a fraction φ of the cash flows is lost every instant of time. This fraction represents the cost of financial distress, for instance, lawyers' charges, administrator's fees, or possibly a loss of possible profitable investment opportunities as managers might be distracted because of their dealings with the creditors.

Hence if the creditors continue then their payoff will be $(\theta x - a)$ per period, where $\theta = 1 - \varphi$. If they liquidate then they get K. If the creditors decide to continue, then they keep their option alive and in the next time period (if the firm stays in default) they again have the option to liquidate. Thus, there is value in waiting. Hence by solving the creditors' dynamic programming problem, we can determine the value of their option which is given by

$$F(x) = \max\left\{K, [\theta x - a]\,dt + \mathsf{E}\left[F(x + dx)\exp(-rdt)\right]\right\}. \tag{3}$$

where E is the expectations operator. Note that the continuation value is given by $C(x) = [\theta x - a]\,dt + \mathsf{E}\left[F(x + dx)\exp(-rdt)\right]$. It is just the payoff which the creditors appropriate in the current period plus the expected present value of the option which they retain. The liquidation point, x^*, is thus determined endogenously. It is the point at which the creditors are indifferent between continuing and liquidating. Hence it is the value of x such that the continuation value just equals the liquidation value. When $x < x^*$, the optimal decision for the creditors is to liquidate.

As shown in Figure 1, the default point differs from the liquidation point. When x falls below $\hat{x}$, default occurs but this need not automatically lead to liquidation. If the firm's condition further deteriorates and its liquidity problems are aggravated then the

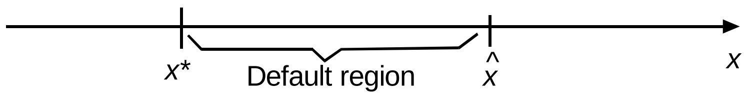

Figure 1: Default and Liquidation. The figure shows how default and liquidation are determined by the state variable. Default occurs when the state variable hits $\hat{x}$. Liquidation occurs when the state variable falls further to x^*.

firm would finally be liquidated by the creditors. Hence liquidation will take place when the state variable falls below x^*.

2.4. Valuation of equity and corporate debt. As long as the firm is in the default region, as depicted in Figure 1, the creditors will have the control rights. However, if the firm manages to overcome its liquidity problems and is restored to a healthy condition, then the control rights are transferred back to the debtors. Thus the creditors are in effect the residual claimants as long as the firm stays in default. Nevertheless as long as the firm is not liquidated there is always the likelihood that the debtors will retain the control rights. Thus for low firm values the creditors act as the residual claimants but for high enough firm values the debtors will be the residual claimants.

Since the control rights can alter from the debtors to the creditors and vice versa, thus the payoffs of both the parties will be state dependent. Furthermore any change in the payoff function will be temporary and reversible until and unless the firm is liquidated. This feature of our model brings us closer to reality and we do not need to rely on payoff functions, the changes of which are permanent and hence irreversible.

As long as the firm is doing well and is not in default, the equityholders will be the residual claimants and their payoffs will be given by $x - a - b$. However in bad states of the word, the equityholders will just earn the value of their outside option and hence their payoff will be zero.[12] The payoff to the equityholders is thus given by

$$e(x) = \begin{cases} x - a - b & \text{if } x \geq \hat{x} \\ 0 & \text{if } x < \hat{x} \end{cases} . \tag{4}$$

Therefore, at any instant the profit flow to the equity holders is given by

$$e(x) = \max\left[x - a - b, 0\right].$$

Notice that the equityholders have a call option on the firm at every instant in time. The option, if exercised at time t means that the equityholders will appropriate the current earnings of the firm in time t given an exercise price of $a + b$. Since each option can only be exercised at that very instant, the equityholders have a number of European call options. However once the state variable hits x^*, the call option of the equityholders becomes worthless as the firm is then liquidated. Hence the value of equity is the sum of the

[12]Note that the value of the owner-manager's outside option is correctly treated as an opportunity cost for the firm.

value of all these call options and it is therefore dependent on the state variable. As x approaches x^*, the value of equity declines. Conversely, the value of equity increases as the earnings of the firm rise.

In this model, equity and debt are time homogenous securities and hence their values are time independent. Using standard no-arbitrage arguments the values of equity and debt can be expressed in terms of the cash flow process. It can be shown that, in this environment, the value of any time independent claim which is a function of the cash flow process, x, will satisfy the following ordinary differential equation

$$\frac{1}{2}\sigma^2 x^2 S''(x) + \alpha x S'(x) - rS(x) + s(x) = 0 \tag{5}$$

where $S(x)$ is the value of security S which is a twice continuously differentiable function of the state variable x, and $s(x)$ is the payoff function of that security.

Thus the value of equity, E, will satisfy the following differential equation

$$\frac{1}{2}\sigma^2 x^2 E''(x) + \alpha x E'(x) - rE(x) + e(x) = 0 \tag{6}$$

where $e(x)$ is as defined in Eqn. (4). To solve for the value of equity we need appropriate boundary equations. Let E_1 be the value of equity when $x \geq \hat{x}$ and E_2 be the value of equity when $x < \hat{x}$. Then the value of equity will obey the following boundary conditions

$$\lim_{x\to\infty} E_1(x) = \frac{x}{r-\alpha} - \frac{a}{r} - \frac{b}{r} \tag{7}$$

$$E_1\left(\hat{x}\right) = E_2\left(\hat{x}\right) \tag{8}$$

$$E_1'\left(\hat{x}\right) = E_2'\left(\hat{x}\right) \tag{9}$$

$$E_2\left(x^*\right) = 0. \tag{10}$$

Eqn. (7) will hold if asset prices are free of speculative bubbles. It just says that as x grows large the value of equity approaches the expected discounted integral of future payoffs which will accrue to the equityholders, i.e. as x increases the value of equity approaches $\mathsf{E}_t \int_t^\infty (x-a-b)\exp[-r(u-t)]du$. Eqn. (8) is the value-matching condition at default. Eqn. (9) says that the slope of the value function of equity has to be continuous at the default point. This follows from Claim (1) as it is optimal for debtors to default only when the firm is facing liquidity problems. Finally, Eqn. (10) is the value-matching condition at liquidation and it says that the value of equity finally falls to zero when the firm is liquidated.

Solving the differential equation (6), subject to the above boundary conditions, we show in the Appendix that we get the following result.

Proposition 1. *Let $E(x)$ denote the value of the levered firm's equity. Assume that the firm has issued perpetual debt with coupon payment b and principle b/r. Suppose that the value of the manager's outside option is given by a. Then if the debt is risky*[13], *i.e. $K < b/r$, the value of equity is given by*

[13] For completeness, in the appendix, we also derive the value of equity for the simpler and less interesting case, when debt is riskless, i.e. when $K \geq b/r$.

$$E(x) = \begin{cases} 0 & \text{if } x < x^* \\ \frac{1}{\hat{x}^{\xi_1} - \hat{x}^{\xi_2} x^{*\xi_1 - \xi_2}} \left[\left\{ \frac{H}{J} \left(\frac{\hat{x}}{r-\alpha} - \frac{\hat{x}}{r} \right) - \frac{1}{J(r-\alpha)} \right\} \hat{x}^{\xi_2} + \frac{\hat{x}}{r-\alpha} - \frac{\hat{x}}{r} \right] & \\ \times \left[x^{\xi_1} - x^{*\xi_1 - \xi_2} x^{\xi_2} \right] & \text{if } x^* \leq x < \hat{x} \\ \frac{x}{r-\alpha} - \frac{a}{r} - \frac{b}{r} + \left[\frac{H}{J} \left(\frac{\hat{x}}{r-\alpha} - \frac{\hat{x}}{r} \right) - \frac{1}{J(r-\alpha)} \right] x^{\xi_2} & \text{if } x \geq \hat{x} \end{cases} \tag{11}$$

where H and J are constants with the following values

$$H = \frac{\xi_1 \hat{x}^{\xi_1 - 1} - \xi_2 x^{*\xi_1 - \xi_2} \hat{x}^{\xi_2 - 1}}{\hat{x}^{\xi_1} - \hat{x}^{\xi_2} x^{*\xi_1 - \xi_2}}$$

$$J = \xi_2 \hat{x}^{\xi_2 - 1} - H \hat{x}^{\xi_2}$$

The powers ξ_1 and ξ_2 are the positive and negative roots respectively of the characteristic quadratic equation $\xi(\xi - 1)\sigma^2/2 + \xi\alpha - r = 0$.

The solution of the equity function from Proposition 1 is illustrated in Figure 2. Note that the value of equity is positive even in the default region even though the economic payoff to the equityholders in this region is zero. This is because as long as the firm is not liquidated, there is always a positive probability that the firm might exit default and hence the equity holders might be able to earn economic profits in the future.

Next we value the corporate debt issued by the firm. The payoff function of bondholders holding risky debt is given by

$$p(x) = \begin{cases} b & \text{for } x \in [\hat{x}, \infty) \\ \theta x - a & \text{for } x \in [x^*, \hat{x}) \\ rK & \text{for } x \in [0, x^*) \end{cases} \tag{12}$$

where x^* is the endogenous point of liquidation chosen by the bondholders. Eqn. (12) states that the bondholders get the full contracted payment in good states of the world. However, in the default region, the bondholders get less than their contracted coupon payment because of the insufficient cash flows generated by the firm. The payoff of the bondholders in the default region is $\theta x - a$ per instant of time, where $\theta = 1 - \varphi$. This is because, as long as the creditors decide to keep the firm alive, they have to pay the manager the value of her outside option, a. Furthermore, in this state of the world, a fraction φ of the firm's cash flows is lost due to financial distress costs. Opler and Titman (1994) find evidence that levered firms incur substantial financial distress costs in industry downturns. These losses might take different forms. They might be customer driven as customers might be reluctant to do business with financially distressed firms. These losses might also be competitor driven if it is the case that financially stronger firms take advantage of their weaker counterparts by aggressive pricing in an attempt to drive out the firms experiencing distressed times. More directly, these costs will also reflect lawyers fees, etc.[14] Finally, the bondholders appropriate the scrap value, K, when the firm is liquidated and their payoff henceforth is rK per instant of time.

[14] The study by Opler and Titman (1994) finds that the costs of financial distress are more pronounced for highly leveraged firms. Thus, we would expect φ to vary with leverage. It will be relatively higher for firms with more debt in their capital structure.

The bondholders' payoff function is piecemeal right continuous and is depicted in panel (a) of Figure 3. Note that it is discontinuous at the points $\hat{x}$ and x^*. In the absence of arbitrage, the value function of debt, $B(x)$ and its first derivative $B'(x)$ must both be continuous at the points $\hat{x}$ and x^*. Thus, the payoff function will be discontinuous if and only if the second derivative of the value function, $B''(x)$ is discontinuous at the points $\hat{x}$ and x^*. (This is clear on examination of the ODE (13)). Dumas (1991) has shown that for optimal stopping problems, smooth-pasting remains a condition only involving the first derivative of claim values and not the second derivative. This therefore would explain the discontinuities in the payoff function.[15] The payoff function of the equityholders (4) analogous to the payoff function of the bondholders will be piecemeal right continuous and for the same reasoning will be discontinuous at the point $\hat{x}$.

The value of debt, $B(x)$, will satisfy the following differential equation

$$\frac{1}{2}\sigma^2 x^2 B''(x) + \alpha x B'(x) - rB(x) + p(x) = 0 \tag{13}$$

where $p(x)$ is as defined in Eqn. (12). To solve for the value of corporate bonds we need to specify appropriate boundary conditions. Let B_1 be the value of debt when $x \geq \hat{x}$. We know that once a default occurs, the bondholders have the option to liquidate, and the value of their option is given by $F(x) = \max\{K, C(x)\}$, where $C(x)$ as defined earlier is the value of debt in the continuation region, i.e. $C(x) = (\theta x - a)\,dt + \mathrm{E}\left[F(x + dx)\exp(-rdt)\right]$. Then the value of corporate debt will satisfy the differential equation (13) subject to the following boundary conditions

$$\lim_{x \to \infty} B_1(x) = \frac{b}{r} \tag{14}$$

$$B_1\left(\hat{x}\right) = C\left(\hat{x}\right) \tag{15}$$

$$B_1'\left(\hat{x}\right) = C'\left(\hat{x}\right) \tag{16}$$

$$C\left(x^*\right) = K \tag{17}$$

$$C'\left(x^*\right) = 0. \tag{18}$$

These boundary conditions have the following interpretations. Eqn. (14) is the no bubbles condition which states that as the state variable approaches higher and higher values, the value of debt approaches the value of risk free debt. In the limit, when the cash flow has an infinite value, the debt of the firm is essentially risk free and its value is just equal to the present value of the coupon payments. Eqn. (15) is the value-matching condition at the default point. Eqn. (16) states that the value of debt must be continuously differentiable across x. Since the Brownian motion of the state variable can diffuse freely across the default boundary, for no arbitrage the claim value cannot change abruptly. Dixit (1993) shows that the if the flow payoff function changes, then the value function should not change abruptly and thus the first derivative of the value function should be

[15] Thus the "super-contact condition" whereby the smooth-pasting condition can be expressed in terms of the second derivative will not apply to our case. See Dumas (1991) for details.

continuous for the absence of arbitrage. (See Karatzas and Shreve (1998) for a more rigorous discussion). Eqn. (17) is the value-matching condition at liquidation. From the Bellman equation (3) we know that in the creditors' stopping region, the value of their option to liquidate is just equal to the scrap value of the firm, and hence by continuity we can impose the value-matching condition (17). Eqn. (18) is the standard smooth-pasting condition which determines the optimal stopping point for the creditors. Thus when the state variable falls below x^* the creditors will find it in their interest to enforce liquidation.

As shown in the Appendix, solving the differential equation (13) given the boundary conditions just described yields the following result.

Proposition 2. Assume that the firm has issued perpetual debt with coupon payment b and principle b/r. Suppose that the value of the manager's outside option is given by a. Suppose that a fraction φ of the firm's cash flows is lost in financial distress costs as long as the firm is in default and that $\theta = 1-\varphi$. Let $B(x)$ denote the value of corporate debt. Then if the debt is risky, i.e. $K < b/r$, the value of debt is given by

$$B(x) = \begin{cases} K & \text{if } x < x^* \\ \frac{\theta x}{r-\alpha} - \frac{a}{r} + \left[\left(\frac{\theta(\xi_2-1)}{(\xi_1-\xi_2)(r-\alpha)}\right)x^{*1-\xi_1} - \frac{\xi_2}{\xi_1-\xi_2}\left[K+\frac{a}{r}\right]x^{*-\xi_1}\right]x^{\xi_1} \\ \quad + \left[\left(\frac{1-\xi_1}{\xi_1-\xi_2}\right)\frac{\theta x^{*1-\xi_2}}{r-\alpha} + \frac{\xi_1}{\xi_1-\xi_2}\left[K+\frac{a}{r}\right]x^{*-\xi_2}\right]x^{\xi_2} & \text{if } x^* \le x < \hat{x} \\ \frac{b}{r} + \left\{\left(\frac{1}{r-\alpha} - \frac{1}{r}\right)\hat{x}^{1-\xi_2} + \left[\left(\frac{\theta(\xi_2-1)}{(\xi_1-\xi_2)(r-\alpha)}\right)x^{*1-\xi_1} - \frac{\xi_2}{\xi_1-\xi_2}\left[K+\frac{a}{r}\right]x^{*-\xi_1}\right]\hat{x}^{\xi_1-\xi_2} \right. \\ \quad \left. + \left[\frac{1-\xi_1}{\xi_1-\xi_2}\frac{\theta x^{*1-\xi_2}}{r-\alpha} + \frac{\xi_1}{\xi_1-\xi_2}\left(K+\frac{a}{r}\right)x^{*-\xi_2}\right]\right\}x^{\xi_2} & \text{if } x \ge \hat{x} \end{cases} . \tag{19}$$

If the debt is riskless, i.e. $K \ge b/r$, then the value of debt is given by $B(x) = b/r$.

While we have derived a closed form solution for corporate debt, the endogenous liquidation point x^* cannot be expressed in closed form. Nevertheless, root finding algorithms can numerically calculate x^*, given the values of other parameters. Given Proposition 2, the Appendix shows that x^* will be the solution to Eqn. (16). Thus the threshold x^* can be computed numerically from Eqn. (16) given other parametric values.

Given the form of the solution, it might be the case that for a certain range of parametric values $x^* < a$. If that happens to be the case, then it would actually imply that the bondholders might be willing to inject cash into the firm if the state variable drops below a/θ. (This can readily be seen from inspection of the payoff function in Eqn. (12)). This might be the case if the scrap value of the firm is low enough relative to a high value of the manager's outside option. Intuition would then suggest that if the manager has a lot of bargaining power during the reorganization process, bondholders might be willing to inject cash so as to retain the manager's human capital as they would realise that liquidation would fetch them a very low value. Nevertheless they will ultimately liquidate if the gap between the state variable, x, and a widens.

The solution from Proposition 2 is depicted in Figure 2. Note that as is the usual case, the value of debt is a concave function of the state variable in good states of the world. However, in the default region the value of debt is a convex function of the state variable reflecting the fact that the bondholders are in effect the residual claimants as long as the firm remains in default. They cannot recoup the full contracted coupon payment but

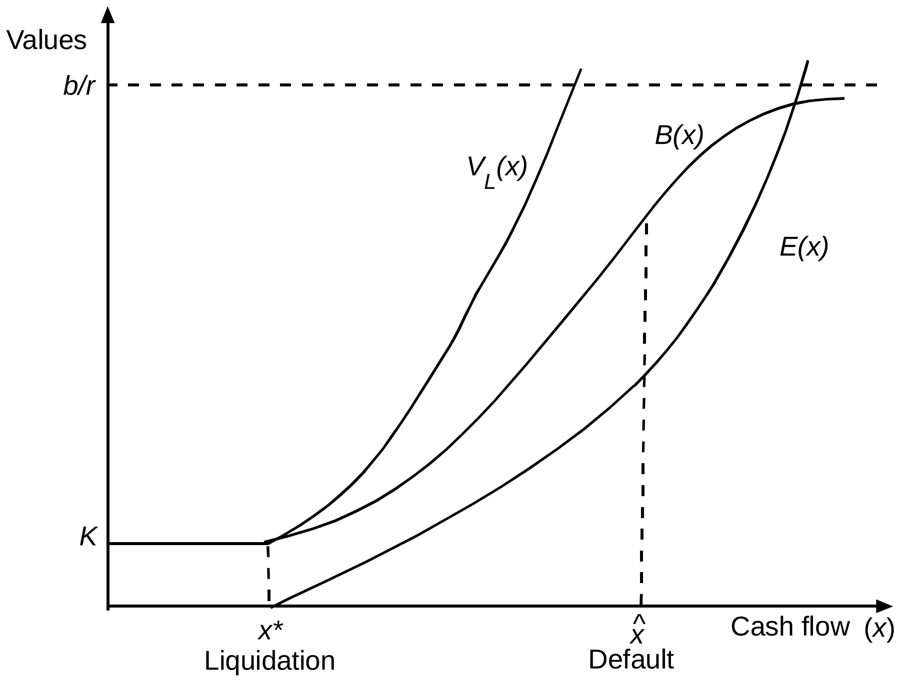

Figure 2: The dynamics of the firm and its securities. The figure shows the value function of the levered firm, $V_L(x)$, and the values of the firm's equity, $E(x)$, and debt, $B(x)$, as functions of cash flow, x.

nevertheless extract as much as is possible after paying off the manager her value of the outside option.

The figure illustrates that default occurs when the state variable first hits $\hat{x}$. However this point might not automatically result in liquidation as creditors might find it in their advantage to retain their option to liquidate. Nevertheless if the financial position of the firm is further aggravated, the creditors might find it in their interest to liquidate the firm. Liquidation will occur when the cash flow falls below the critical level x^*. Thus the liquidation point will in general differ from the point of default.

3. Does the Modigliani-Miller Theorem hold?

Modigliani and Miller (1958) showed that the market value of any firm is independent of its capital structure and thus the value of a levered firm should be the same as the value of an unlevered firm. However, we show that given our set-up, capital structure irrelevance in general will not be obtained even in the absence of financial distress costs and taxes. We first value an unlevered firm in the same set-up as before and then we provide an explanation as to why the value of the unlevered firm would in general differ from the value of its levered counterpart.

3.1. The Value of an Unlevered firm. We now consider the valuation of a pure equity firm. Apart from no debt in the capital structure, the set-up is essentially the same as described in Section 2.1. The owner-manager running the firm can always shut down the firm for a scrap value of K and take her outside option a. Given a competitive labour market, a will also be the minimum level of consumption required by the manager every period in time. Thus the owner-manager in effect has a put option to liquidate the firm, where the exercise price of the option is K.

The value of the owner's put option, $G(x)$, is given by

$$G(x) = \max\{K, (x-a)\,dt + \mathsf{E}[F(x+dx)\exp(-rdt)]\}. \tag{20}$$

Hence the manager in every instant of time, has an option to shut down the firm and fetch a value of K or to continue and get a net payoff of $x - a$ in the current period plus retain the option to liquidate in future periods. The point of liquidation is given by

$$L^* = \max(a, x_u^*) \tag{21}$$

where x_u^* is the value of the cash flows such that the scrap value just equals the continuation value. The constraint (21) is imposed as the manager needs to consume a minimum amount of a per period.[16] Hence the liquidation point will be the higher of the value of the manager's outside option and the critical level of the state variable such that the continuation value in Eqn. (20) just equals the scrap value.

The payoffs of the owner-manager are given by

$$v(x) = \begin{cases} x-a & \text{if } x \geq L^* \\ rK & \text{if } x < L^* \end{cases} \tag{22}$$

where L^* is the point of liquidation chosen by the manager as described above.

The value of the unlevered firm, V_u, will satisfy the following differential equation

$$\frac{1}{2}\sigma^2 x^2 V_u''(x) + \alpha x V_u'(x) - rV_u(x) + v(x) = 0 \tag{23}$$

where $v(x)$ is as defined in Eqn. (22). To solve for the value of the unlevered firm, appropriate boundary conditions are required. In the case of the unlevered firm, we just have three boundary conditions, given that we do not have the issue of default here. The boundary conditions are as follows

$$\lim_{x\to\infty} V_u(x) = \frac{x}{r-\alpha} - \frac{a}{r} \tag{24}$$

$$V_u(x_u^*) = K \tag{25}$$

$$V_u'(x_u^*) = 0. \tag{26}$$

These boundary conditions have straightforward interpretations. Eqn. (24) is the no bubbles condition. Eqns. (25) and (26) are the value-matching and smooth-pasting conditions respectively.

We show in the Appendix that solving for the differential equation (23) subject to the boundary conditions (24),(25) and (26), the following result is obtained.

[16] Even though the manager will always be better off by stopping at x_{u}^* rather than a, nevertheless if $a > x_{\mathsf{u}}^*$ the manager will have to shut down when the state variable drops below a. This will be the case given that the manager has limited wealth and cannot inject further cash into the firm.

Proposition 3. Let $V_u(x)$ denote the total value of the unlevered firm. Assume that the value of the manager's outside option is given by a. Then the value of the unlevered firm is given by

$$V_u(x) = \begin{cases} K & \text{if } x < L^* \\ \frac{x}{r-\alpha} - \frac{a}{r} + \left[\left(K + \frac{a}{r}\right) x^{*-\xi_2} - \frac{x^{*1-\xi_2}}{r-\alpha}\right] x^{\xi_2} & \text{if } x \geq L^* \end{cases} \tag{27}$$

where

$$L^* = \max(a, x_u^*)$$

and

$$x_u^* = \left[K + \frac{a}{r}\right] \left(\frac{\xi_2(r-\alpha)}{\xi_2 - 1}\right). \tag{28}$$

Here, ξ_2 is the negative root of the characteristic quadratic equation $\xi(\xi-1)\sigma^2/2+\xi\alpha-r = 0$.

Note that now we are able to find a closed form expression for the liquidation point given the relatively simple nature of the valuation of the unlevered firm in this set-up. For a certain range of parametric values it will be the case that $x_u^* < a$. In fact, it can be shown that this will be the case if and only if $K < a\lambda$, where the parameter λ is a constant such that $\lambda \equiv [(\xi_2 - 1)/\xi_2 (r-\alpha) - 1/r]$. The intuition behind this is that if the scrap value is very low compared to the opportunity cost of staying in the business, then the manager would actually prefer to inject some money and keep the firm alive as long as the state variable is above x_u^*. However as discussed in footnote 16, the manager has limited wealth and cannot inject further cash into the firm. Further, the manager consumes at least a per period. Given this resource constraint, the manager will have to follow the liquidation rule as defined in Eqn. (21). As discussed in Section 3.2, this wealth constraint can actually be a potential source of inefficiency for the unlevered firm.

Before discussing the efficiency of the unlevered firm vis-a-vis the levered firm, it is useful to restate the above fact in the following condition.

Condition 1. $L^* = a$ if and only if $K < a\lambda$, where $\lambda \equiv [(\xi_2 - 1)/\xi_2 (r-\alpha) - 1/r]$.

Condition 1 states that the unlevered firm will be liquidated at the point where the state variable hits the value of the manager's outside option if and only if the scrap value of the firm is very low relative to the opportunity cost of continuing the firm.

3.2. First Best Liquidation, inefficiencies, and Capital Structure Relevance. We now turn to the interesting issue of whether the value of the unlevered firm is higher compared to its lever counterpart. To address this issue we first need a benchmark for efficiency. The values of the levered and unlevered firms will differ if their liquidation points differ. Further, a measure of the inefficiency of the firm would be the magnitude of the difference of its liquidation point relative to the first best liquidation point.

Given our analysis, it is straightforward to identify the first best liquidation point of a firm. By definition, the first best liquidation point of a firm is the point which maximises the value of the firm. For the unlevered firm, the first best liquidation point also maximises the value of equity. As discussed in Section 3.1, the value of equity will be maximised if the manager follows the liquidation rule such that she liquidates whenever the cash flows fall below x_u^*. Nevertheless, as discussed earlier, the manager might have to liquidate earlier given her wealth constraint, even though she would have been better off if liquidation had occurred at x_u^*. Thus, x_u^* is the optimal liquidation point as it is identified by the smooth-pasting condition, which itself by definition is an optimality condition. Thus we know that the value of the unlevered firm will be at a maximum if the manager follows the liquidation rule such that she liquidates when the state variable hits x_u^*. We have thus proved the following Proposition.

Proposition 4. The optimal liquidation rule, which maximises the value of the firm, is to liquidate the firm when the state variable falls below x_u^*. Thus the first best liquidation point is given by

$$x_u^* = \left[K + \frac{a}{r}\right]\left(\frac{\xi_2(r-\alpha)}{\xi_2 - 1}\right)$$

Having identified the first best efficient liquidation rule, we next turn to the question of whether the Modigliani-Miller Theorem holds and if not then whether the value of the unlevered firm is necessarily higher than the value of the unlevered firm. From inspection of Proposition 3 it is immediately clear that the value of the unlevered firm is not equal to the sum of the value of equity and the value of debt of the levered firm. (The values of the levered firm's equity and debt are given by Propositions 1 and 2 respectively.) This is true even if the financial distress costs are set to zero. Therefore, an important corollary to Proposition 3 is

Corollary 1. The Modigliani-Miller Theorem does not hold even in the absence of financial distress costs and taxes.

The reasoning behind the capital structure relevance result obtained is subtle but naturally follows from our model. Capital structure irrelevance will not be obtained if the liquidation point of the levered firm differs from the liquidation point of the unlevered firm. For now, lets suppose that Condition 1 is not satisfied and thus $x_u^* > a$. Thus the manager in the unlevered firm follows the first best liquidation rule, and liquidates when the state variable hits x_u^* as defined in Eqn. (28). Corollary 1 therefore implies that the creditors' liquidation point will differ from the liquidation point of the manager in the unlevered firm, i.e. $x^* \neq x_u^*$. We now provide the intuition behind this.

In the levered firm, once a default occurs the creditors have the option to liquidate the firm. In the unlevered firm, the manager always has an option to liquidate the firm for a scrap value of K. In fact it will be the case that the creditors will exercise their (put) option to liquidate earlier vis-a-vis the manager in the unlevered firm even in the absence of financial distress costs. Figure 3 compares the payoff function of the creditors and of the manager in the unlevered firm. For simplicity assume that there are no financial distress costs, i.e. $\theta = 1$. Then note that both the creditors and the manager have the same payoff function for low fundamental values of the firm. In bad states of the world, the creditors are the residual claimants and their payoff is thus $x - a$. In the unlevered firm, the manager always is the residual claimant with a payoff of $x - a$ as long as the firm is not liquidated. Thus in good states of the world, while the manager is still the residual claimant and earning $x - a$, the creditors payoff will be capped by the amount of the coupon payment b.

Thus both the creditors and the unlevered firm manager have put options to liquidate but these options have different values as the payoff functions of the two parties differ. In

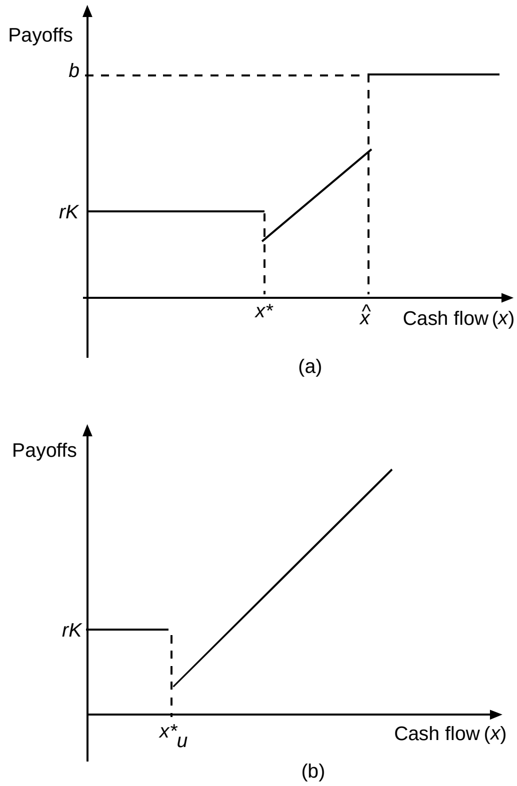

Figure 3: The payoff functions. Panel (a) of the figure depicts the payoff function of the bondholders in the levered firm while panel (b) depicts the payoff function of the owner manager in the unlevered firm. Both of them have options to liquidate, but in the absence of any wealth constraints, the bondholders will liquidate earlier given their capped payoff. Because of this "capped payoff" effect, the value of the levered firm will be lower than the value of the unlevered firm in the absence of any wealth constraints of the manager in the unlevered firm.

fact the creditors put option is less valuable due to an upper limit to their payoffs. The creditors realise that their payoffs are capped by the amount of their coupon payment and they take this into account when determining the liquidation point. Because of the upper limit to the creditors' payoff, they would liquidate earlier compared to the manager in the unlevered firm. It will therefore always be the case that $x^* > x_u^*$. We thus have the following proposition.

Proposition 5. The creditors will always liquidate the firm prematurely relative to the first best liquidation point.

Thus if Condition 1 is not satisfied it will always be the case that the creditors will liquidate earlier compared to the unlevered manager and thus the value of the unlevered firm will be higher than the value of the levered firm, given the inefficiency induced in the levered firm by the creditors' inefficient liquidation decision. The creditors liquidation decision would maximise the value of their claim but not the value of the firm as a whole.

Note that the Coase theorem will fail in this environment given the limited wealth of the manager. Thus both the parties will be unable to bargain their way towards the efficient outcome.

More generally, depending on the parametric values, the following three cases might arise.

Case 1: $a < x_u^* < x^*$. In this case, Condition 1 is not satisfied and thus $L^* = x_u^*$. Thus now the wealth constraint of the managers in the unlevered firm is not binding and hence they will follow the first best liquidation rule as defined in Proposition 4. But since $x_u^* < x^*$, the creditors in the levered firm will liquidate too early and thus the value of the levered firm will be less than the value of the unlevered firm.

Case 2: $x_u^* < a < x^*$. Note that now since $x_u^* < a$, Condition 1 holds and thus the wealth constraint of the manager in the unlevered firm is binding. Therefore the manager of the unlevered firm will now liquidate when the state variable hits a and thus given her resource constraint, the manager will be unable to follow the first best liquidation rule. Nevertheless the value of the unlevered firm will still be higher than the value of the levered firm given that the liquidation rule followed by the creditors in the levered firm is even more inefficient relative to the liquidation rule being followed by the manager in the unlevered firm. More Formally, $|L^* - x_u^*| < |x^* - x_u^*|$ and thus the deviation from the first best liquidation rule is higher in the case of the levered firm.

Case 3: $x_u^* < x^* < a$. Again, as in Case 2, Condition 1 holds and thus the resource constraint of the manager in the unlevered firm is binding. Thus the liquidation point of the unlevered firm's manager is now given by $L^* = a$. However one critical difference between Case 1 and Case 2 is that the resource constraint is now binding at a level which is higher than the liquidation point chosen by the creditor in the levered firm. The implication is that now $|L^* - x_u^*| > |x^* - x_u^*|$ and hence now the deviation from the first best level is actually higher for the unlevered firm. Thus the unlevered firm will be more inefficient, in terms of its liquidation threshold, relative to the levered firm. Thus, given Case 3, the value of the levered firm will actually be higher than the value of the unlevered firm.

We have now identified all possible cases that might arise. Our general result is that the value of the levered firm will in general be not equal to the value of the unlevered firm. Furthermore, the levered firm would never achieve the first best given an upper limit to the creditors' payoff. Nevertheless the unlevered firm might be able to achieve the first best if the wealth constraint of the manager is not binding. However, if the wealth constraint of the manager is binding, then the value of the unlevered firm may or may not be higher than the value of the levered firm. Intuitively, the value of the unlevered firm might happen to be lower than that of the levered firm if equityholders do not have enough resources to achieve the first best level.[17]

4. Discussion

The main objectives of our study were to construct a dynamic debt pricing model that comes close to the structure of most existing bankruptcy procedures; to explicitly incorporate the inalienability of the human capital which is increasingly a key feature of the modern enterprise; and lastly but not least importantly to provide a clear distinction between default and liquidation.

[17] Note that Case 3 will arise however for only a very low liquidation value relative to a high value of the manager's outside option.

The importance of the last objective cannot be overemphasized. It is clear that most of the companies who default go into a period of reorganization and may or may not be liquidated. The time spent in the reorganization period varies immensely. Franks and Torous (1989) report that in their sample, firms on average spend a period of 4 years in Chapter 11. Gilson, John and Lang (1990) find that in their sample only about 5% of the bankruptcies in Chapter 11 are converted into Chapter 7 liquidations. They report that this proportion varies from sample to sample and other studies have found that about one-third of the firms in Chapter 11 end up in liquidations. Franks and Sussman (2000) in their study on distressed UK companies find that banks use their control rights to encourage financially distressed firms to restructure. They find no evidence of automatic liquidation upon default. There are numerous examples of companies who pass through a period of reorganization after default but come out successful. It would therefore be incorrect to treat default and liquidation as being synonymous when modelling the levered firm.

Our study does not only provide an arbitrage-free model for valuing equity and corporate debt but also provides a framework to infer some interesting corporate finance implications. In our framework both debtors and creditors can at any point in time be the residual claimants. This is remarkably in contrast to the traditional "nexus of contracts" definition of the firm first provided by Jensen and Meckling (1976). As argued by Rajan and Zingales (2000), in a world of incomplete contracts and multiple sources of power no party has predetermined payoffs and any party can at some point in time have a residual claim on the firm's assets. This explains why maximisation of the value of the residual claimant's securities need not maximise the value of the firm as a whole.

As argued by Jensen (1986) one motivation for issuing debt is to commit the managers of the firm to pay out future cash flows. However for such an objective to be effective there has to be some punishment if the commitment is not fulfilled. This punishment can take various forms. In our model it is the shift of the control rights that gives force to this commitment. Managers realise that if a circumstance arises where they are unable to service the debt, then they will be left with a gross payoff that just equals the value of their outside option.

Our framework also identifies the first best liquidation threshold which can then be used as a benchmark to compare the efficiency of the unlevered firm versus the levered firm. We find that the Modigliani-Miller theorem does not hold even in the absence of any coordination problems, informational asymmetries, financial distress costs and taxes. Our study finds that in general the value of the levered firm will be less than the value of the all equity firm as the creditors have an incentive to liquidate prematurely given an upper limit to their payoffs. However depending on the parametric values, a situation might arise where the value of the levered firm is greater than the value of the all equity firm. The intuition underlying this is that the all equity firm might be cash constrained because of which the equityholders might have to liquidate earlier than they would have wanted to if their resource constraint had not been binding. Note that since the parametric values will differ from firm to firm and industry to industry, therefore, the efficiency of the liquidation thresholds vis-a-vis the first best liquidation point will also differ on a case by case basis.

5. Summary and Conclusions

We have set up a debt pricing model with the following features: (a) the structure of our model closely resembles the existing bankruptcy regimes; (b) the model incorporates the inalienability of human capital which is increasingly becoming a dominant feature of the modern firm; (c) a clear separation between default and liquidation is obtained; (d) the liquidation decision is made by the creditors following a default by the manager; (e) capital structure irrelevance not obtained even in the absence of financial distress costs and taxes; (f) provides a framework to compare the efficiency of the liquidation thresholds of different firms. We have therefore provided a framework which is rich enough for valuing corporate securities and inferring some useful implications for corporate finance.

APPENDIX

Proof of Proposition 1: The technical problem one faces here is that of solving the ordinary differential equation (6) subject to the boundary conditions (7), (8), (9) and (10). If $x < \hat{x}$, then $e(x) = 0$ and Eqn. (6) simply becomes the following second order homogenous differential equation

$$\frac{1}{2}\sigma^2 x^2 E_2''(x) + \alpha x E_2'(x) - rE_2(x) = 0 \tag{29}$$

where E_2 is the value of equity when $x < \hat{x}$. It is easy to verify that the general solution of Eqn. (29) is

$$E_2(x) = N_1 x^{\xi_1} + N_2 x^{\xi_2} \tag{30}$$

where N_1 and N_2 are the two integration constants and ξ_1 and ξ_2 are the positive and negative roots respectively of the characteristic equation $\xi(\xi-1)\sigma^2/2 + \xi\alpha - r = 0$. If $x \geq \hat{x}$, then $e(x) = x - a - b$ and the differential equation becomes

$$\frac{1}{2}\sigma^2 x^2 E_1''(x) + \alpha x E_1'(x) - rE_1(x) + x - a - b = 0 \tag{31}$$

where $E_1$is the value of equity when $x \geq \hat{x}$. The general solution to Eqn. (31) is

$$E_1(x) = M_1 x^{\xi_1} + M_2 x^{\xi_2} + \frac{x}{r-\alpha} - \frac{a}{r} - \frac{b}{r} \tag{32}$$

where $x/(r-\alpha) - a/r - b/r$ is the particular solution and M_1 and M_2 are the integration constants. As $x \to \infty$, x^{ξ_1} explodes. Thus, given the no bubbles condition (7), M_1 must be zero. We know that the value of equity is zero when x falls below x^*. We therefore need to determine the value of equity in the ranges $x \in [x^*, \hat{x})$ and $x \in [\hat{x}, \infty)$ which is given by E_2 and E_1 respectively. We therefore now have three unknowns N_1, N_2 and M_2 in Eqns. (30) and (32) and three equations given by the boundary conditions (8), (9) and (10). Solving for the three unknowns, yields the solution in Eqn. (11). Q.E.D.

Value of equity with riskless debt: If $K \geq b/r$, then it will always be in the interest of the creditors to liquidate once a default occurs as they can then recover the total amount of their principal. The default point is therefore the same as the liquidation point in this case. Further the equityholders will be the residual claimants in all states of the world. Their payoffs are now given by

$$e(x) = \begin{cases} x - a - b & \text{if } x \geq \hat{x} \\ rK - b & \text{if } x < \hat{x} \end{cases} . \tag{33}$$

It is obvious that when debt is riskless the value of equity when $x < \hat{x}$ is given by $K - b/r$. We therefore need to determine the value of equity when $x \geq \hat{x}$. The value of equity when $x \geq \hat{x}$ will satisfy the following second order differential equation

$$\frac{1}{2}\sigma^2 x^2 E''(x) + \alpha x E'(x) - rE(x) + x - a - b = 0.$$

The general solution of this equation is given by

$$E(x) = Q_1 x^{\xi_1} + Q_2 x^{\xi_2} + \frac{x}{r-\alpha} - \frac{a}{r} - \frac{b}{r}$$

where Q_1 and Q_2 are the constants to be determined. The two boundary conditions that we need are

$$\lim_{x\to\infty} E(x) = \frac{x}{r-\alpha} - \frac{a}{r} - \frac{b}{r} \tag{34}$$

$$E\left(\hat{x}\right) = K - \frac{b}{r} \tag{35}$$

The asymptotic no bubbles condition implies that $Q_1 = 0$. Therefore we now have one equation in one unknown. This readily yields the following solution for the value of equity with riskless debt

$$E(x) = \begin{cases} K - \frac{b}{r} & \text{if } x < \hat{x} \\ \frac{x - x^{1-\xi_2}}{r-\alpha} - \frac{a}{r} - \frac{b}{r} + \left(K + \frac{a}{r}\right) x^{-\xi_2} & \text{if } x \geq \hat{x} \end{cases}. \tag{36}$$

Proof of Proposition 2: Here we want to solve Eqn. (13) subject to boundary conditions (14), (15), (16), (17) and (18). When $K < b/r$, the value of debt is just equal to K for $x < x^*$, as when the firm is liquidated the maximum that the debtholders can get is the scrap value of the firm. Let the value of debt be $C(x)$ for $x \in [x^*, \hat{x})$. Then given the payoff function in Eqn. (12), Eqn. (13) becomes

$$\frac{1}{2}\sigma^2 x^2 C''(x) + \alpha x C'(x) - rC(x) + \theta x - a = 0. \tag{37}$$

It is easy to verify that the general solution to Eqn. (37) is of the following form

$$C(x) = A_1 x^{\xi_1} + A_2 x^{\xi_2} + \frac{\theta x}{r-\alpha} - \frac{a}{r} \tag{38}$$

where A_1 and A_2 are the two integration constants. Next consider the range $x \in [\hat{x}, \infty)$. Let B_1 be the value of debt in this range. Now Eqn. (13) becomes

$$\frac{1}{2}\sigma^2 x^2 B_1''(x) + \alpha x B_1'(x) - rB_1(x) + b = 0. \tag{39}$$

The general solution to Eqn. (39) is

$$B_1(x) = D_1 x^{\xi_1} + D_2 x^{\xi_2} + \frac{b}{r} \tag{40}$$

where D_1 and D_2 are the integration constants. The asymptotic condition (14) implies that the coefficient D_1 will be zero as ξ_1 is positive. Therefore we now have four equations (15), (16), (17), (18) in four unknowns, A_1, A_2, D_2 and x^*. Note that the free boundary x^* itself is an unknown. We solve for the value of debt in terms of x^*. Using Eqns. (15), (17) and (18) we solve for the unknown constants A_1, A_2 and D_2 in terms of x^*. This yields the expression for $B(x)$ given in Eqn. (19). Note that we are still left with one unknown x^* and one equation (16). However closer examination of Eqn. (16) reveals that it is nonlinear in the threshold x^*. Thus x^* can only be solved numerically from Eqn. (16). Finally, if $K \geq b/r$, then debt is riskless and hence there is no default risk. Thus the value of debt now just equals its principal value of b/r. Q.E.D.

Proof of Proposition 3: Again we need to solve an ordinary differential equation subject to boundary conditions. The ODE is given in Eqn. (23) and the boundary conditions are stated in Eqns. (24), (25) and (26). We know that the value of the unlevered firm will be equal to K if $x < L^*$, where L^* is defined in Eqn. (21). If $x \geq L^*$, then given the payoff function in Eqn. (22), the ODE becomes

$$\frac{1}{2}\sigma^2 x^2 V_u''(x) + \alpha x V_u'(x) - r V_u(x) + x - a = 0 \tag{41}$$

the general solution of which is given by

$$V_u(x) = R_1 x^{\xi_1} + R_2 x^{\xi_2} + \frac{x}{r-\alpha} - \frac{a}{r} \tag{42}$$

where R_1 and R_2 are the constants to be determined. Given the asymptotic no bubbles condition we immediately know that R_1 equals zero. We are now left with two equations (25), (26) in two unknowns R_2 and x_u^*. Recall that x_u^* is a free boundary which determines L^* and which itself needs to be determined. Solving the two equations for the two unknowns yields the solution given in Eqns. (27) and (28). Q.E.D.

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