Risk Management With Default-risky Forwards

Olaf Korn*

Abstract

This paper studies the impact of counter-party default risk of forward contracts on a firm´s production and hedging decisions. Using a model of a risk-averse competitive firm under price uncertainty, it derives several fundamental results. If expected profits from forward contracts are zero, the hedge ratio is surprisingly not affected by default risk under general preferences and general price distributions. This robustness result still holds if forwards are subject to additional basis risk. In general, the analysis shows that default risk of forward contracts is no valid reason to reduce hedge ratios. However, a firm´s optimal output is negatively affected by default risk and it is generally advisable to hedge default risk with credit derivatives.

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* Assistant Professor, Finance Group, University of Mannheim, D-68131 Mannheim, Phone: +49 621 181 1487, Fax: +49 621 181 1519, E-mail: o.korn@uni-mannheim.de. Financial support from the Deutsche Forschungsgemeinschaft (DFG) is gratefully acknowledged.

1 Introduction

The risk that a counterparty defaults is negligible for most exchange-traded derivatives. However, default risk can be substantial for contracts traded on over-the-counter (OTC) markets, like forwards and swaps. Because OTC derivatives markets have grown rapidly and are particularly important for exchange rate and interest rate risk management, the issue of counter-party default risk can not be ignored.

Two problems that are connected with derivatives´ default risk have dominated the literature. The first one concerns the systemic risk caused by potential defaults of derivatives contracts and the related issue of regulation of OTC markets. (See e.g. Hentschel and Smith (1994) and Schachter (1997)). The second problem concerns the valuation of derivatives contracts subject to default (See e.g. Johnson and Stulz (1987), Hull and White (1995), Jarrow and Turnbull (1995), and Duffie and Huang (1996)).

This paper studies a third important problem. What is the effect of counter-party default risk on a firm´s risk management strategy? Survey results show that many non-financial firms using derivatives contracts have major concerns about counter-party default risk (See Bodnar, Hayt, and Marston (1996, 1998)). However, whether and how non-financial firms should adjust their derivatives positions in response to default risk is an open question. This paper provides a first analysis of the issue.[1]

The analysis makes use of an extended version of Holthausen´s (1979) model of a competitive firm under price uncertainty. In this model framework, the firm is risk averse and simultaneously determines its output and its forward position. However, in contrast to Holthausen´s assumption, forward contracts are subject to default. It is shown how default risk affects the optimal hedge ratio, the optimal output quantity and the interplay between production and hedging decisions. Model extensions explore the effects of credit derivatives written on forward contracts and of additional risk factors.

The paper derives an interesting robustness result: If the expected profit of forward

[1] Many papers have addressed the question of how firms should design risk management strategies with derivatives, but these papers generally assume that derivatives contracts are default free. See e.g. Anderson and Danthine (1980, 1981), Moschini and Lapan (1995), Adam-Müller (1997), Chowdhry and Howe (1999), and Brown and Toft (2002).

contracts is zero, default risk does not affect the hedge ratio. This result holds for general concave utility functions and general distributions of the price risk. Even with additional basis risk, there is no hedge ratio effect. However, default risk reduces a firm´s optimal output.

2 The Basic Model

Consider a firm that produces a single good. The firm makes the decision about the output quantity Q at time 0. At time 1, the good is sold in a competitive market for a price $\tilde{P}_1$, which is a random variable from the perspective of time 0. The firm´s production costs are described by a cost function $c(Q)$, that is increasing, strictly convex and twice differentiable.

Forward contracts are available to the firm as an additional instrument to change the distribution of its profits. These contracts have an exogenous forward price F_0 and can be entered into at time 0. They are written on the price $\tilde{P}_1$ and expire at time 1. Either long positions or short positions in forwards are allowed, and h denotes the number of contracts sold.

Up to this point, the setting is identical to the one of Holthausen´s (1979) model. The crucial difference lies in the possible default of forward contracts when they expire at time 1. In the case of default, forward contracts become asymmetric instruments. If the value of the forward contracts at expiration is negative, the firm has to fulfill its obligations to the contracts. However, if the value of the forward contracts is positive, and it is the case that the counterparty has defaulted, the firm looses all profits from the forward contracts.[2]

Under the above assumptions, the firm´s profit for the period from time 0 to time 1 is equal to

$$\tilde{\Pi} = \tilde{P}_1 Q - c(Q) + h(F_0 - \tilde{P}_1) - \tilde{I} \max\left[h(F_0 - \tilde{P}_1), 0\right]. \tag{1}$$

On the right hand side of equation (1), $\tilde{P}_1 Q$ provides the total revenues from selling the good and c(Q) provides the total costs of producing it. The profit or loss from

[2]The assumption of a complete loss is relaxed in Subsection *3.3*, which presents a model extension with a stochastic recovery rate.

selling h units of default-free forward contracts equals $h(F_0 - \tilde{P}_1)$. A possible default of forward contracts shows up in the last term, $-\tilde{I} \max \left[h(F_0 - \tilde{P}_1), 0\right]$. Here, the Bernoulli(p) distributed random variable $\tilde{I}$ indicates whether the counterparty of the forward contract has defaulted or not. With probability $1 - p$, there is no default on the forward contract ($I = 0$), and the forward shows the same payoff as a default-free contract. With probability p, there is a default ($I = 1$), and the firm looses all profits from the forward contracts. Note that $\tilde{P}_1$ and $\tilde{I}$ can in general be stochastically dependent random variables.

The firm solves the following maximization problem:

$$\max_{Q,h} E[U(\tilde{\Pi})], \quad \text{s.t. } \tilde{\Pi} \text{ as defined in equation (1)}, \tag{2}$$

where $U(\Pi)$ denotes a von Neumann - Morgenstern utility function with $U' > 0$ and $U'' < 0$, i.e., the firm is risk averse.[3] To solve the maximization problem (2), the optimal output quantity $Q^\star$ and the optimal forward position $h^\star$ must be determined. In the following, it is assumed that $Q^\star$ is positive.

2.1 Optimal Hedge Ratio

A well known result for default-free forward contracts states that firms should fully hedge their price exposure if and only if forward contracts earn an expected profit of zero, i.e., the hedge ratio $h^\star/Q^\star$ should be equal to one (See e.g. Holthausen (1979) and Feder, Just and Schmitz (1980)). If the expected profit from selling forward contracts is positive (negative), i.e., there is an additional speculative component in the firm´s demand for forwards, then the hedge ratio should be greater than one (smaller than one). The following proposition states that the same result holds for default-risky forwards.

Proposition 1: *If and only if the expected profit from selling forward contracts is positive (zero)(negative), the optimal hedge ratio $h^\star/Q^\star$ is greater than one (equal to one)(smaller than one), i.e.,* $F_0 \gtreqless E[\tilde{P}_1] + E\left[\tilde{I} \max[(F_0 - \tilde{P}_1), 0]\right] \Leftrightarrow h^\star/Q^\star \gtreqless 1$.

[3]The literature has identified many reasons why firms might be risk averse, e.g. taxes, deadweight costs associated with financial distress, agency problems, asymmetric information, and the inability of owners or managers to diversify. See e.g. Stulz (1984), Smith and Stulz (1985), Bessembinder (1991), Froot, Scharfstein and Stein (1993), and DeMarzo and Duffie (1995).

Formal proofs of propositions are generally deferred to the appendix. However, to highlight the intuition behind the result, assume that the expected profit of forward contracts is zero, $Q^\star = 1$, and the hedge ratio equals $h/Q^\star = 1$. Given these assumptions, figure 1 depicts the profit Π and the payoff of a sold forward contract as a function of the price P_1. The upper part of the figure refers to the situation when the forward contract does not default. In this case the profit is a constant function of P_1, i.e., a full hedge completely eliminates the price risk of profits. If expected profits from forward contracts are zero, eliminating risk is the optimal strategy for all risk averse firms. The lower part of figure 1 shows an analogous graph for the case of a defaulting forward contract. If the forward defaults, the firm´s profit is no longer a constant function of P_1, but increases with P_1 for $P_1 < F_0$. The firm faces a price risk. However, if forwards default and $P_1 < F_0$, the firm´s profit does not depend on the number of forward contracts taken, since the payoff of a forward is always zero. In such a situation, the best thing a risk averse firm can achieve with forward contracts is to equalize profits in all states of nature where forward contracts have an impact on profits. As can be seen from figure 1, this is exactly what a hedge ratio of $h/Q^\star = 1$ does.

Proposition 1 provides an interesting robustness result. If forward contracts earn an expected profit of zero[4], the hedge ratio is the same for default-free and default-risky forwards. In particular, the hedge ratio depends neither on the default probability p, nor the distribution of the price $\tilde{P}_1$, nor the specific form of the utility function. The full hedge is also the variance minimizing hedge. Thus, proposition 1 highlights the importance of a variance minimizing hedge even under general preferences.

Proposition 1 does not state that default risk generally has no impact on the hedge ratio. If the expected profit of forward contracts is non-zero, a speculative component comes into play. In this case, the sign of the firm´s extra demand on forward contracts is the same for default-free and default-risky contracts. However, the exact value of the hedge ratio could be different.

[4]Whether it is reasonable for a firm to assume an expected profit of zero depends ultimately on the firm´s information set. Recent empirical studies on the expectations hypothesis in foreign exchange markets show that it is difficult to reject this hypothesis statistically. See e.g. Bekaert and Hodrick (2000), Roll and Yan (2000), and Maynard and Phillips (2001).

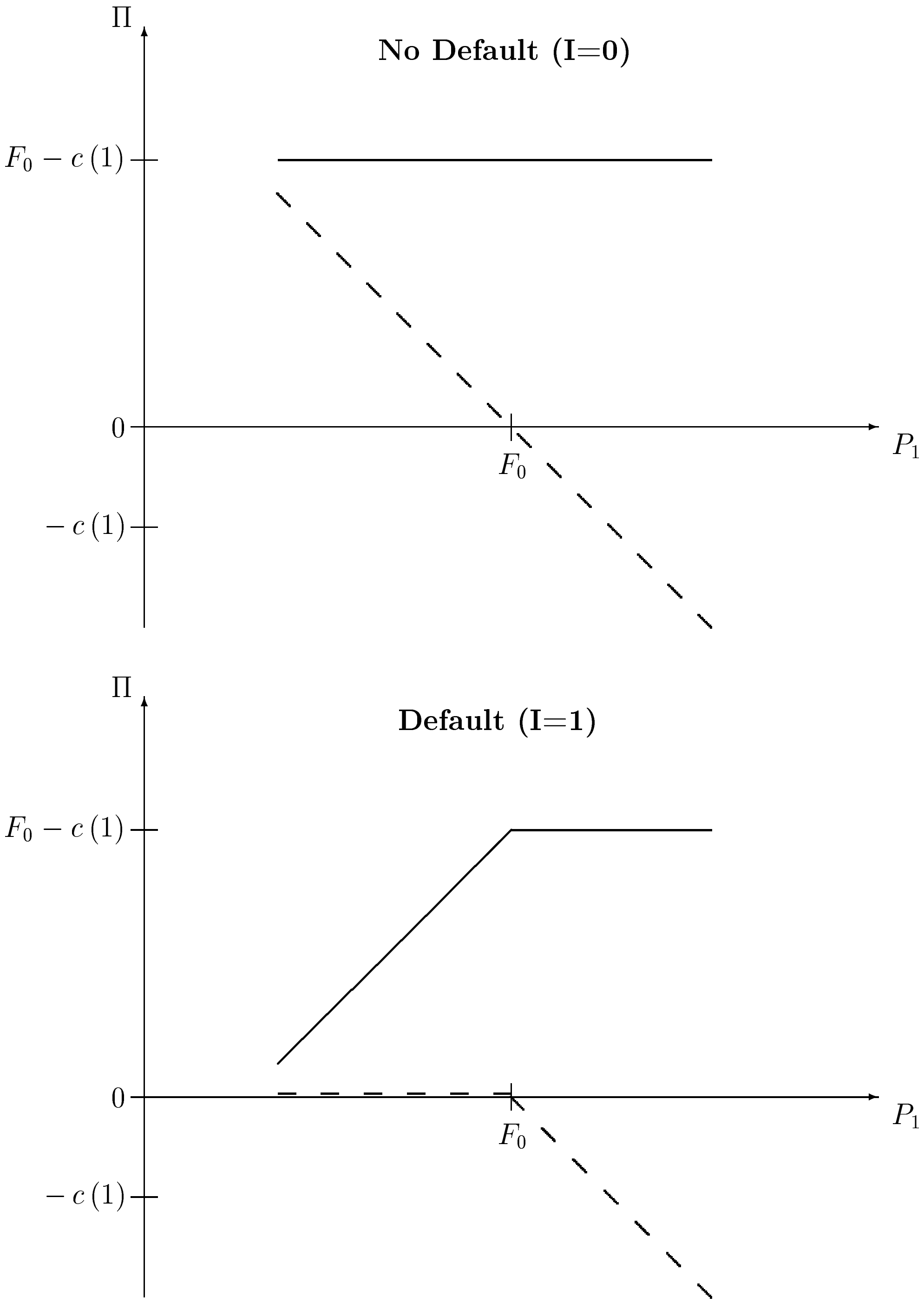

Figure 1: Firm´s profit and payoff of a sold forward contract as functions of P_1

The figure shows the firm´s profit (solid lines) and the payoff of a sold forward contract (dashed lines) as functions of the price P_1. The upper part refers to the case when the forward does not default, the lower part refers to the case when the forward does default. The figure assumes that $Q^\star = 1$ and $h/Q^\star = 1$.

2.2 Optimal Output

If output is endogenous, default risk of forward contracts has another effect. With default-free forwards, the marginal production costs equal the forward price at the optimal output level, i.e., $c'(Q^\star) = F_0$ (See e.g. Holthausen (1979) and Feder, Just and Schmitz (1980)). This well known result identifies the forward price as the focal point of a firm´s production decision, not the future spot price or its distribution. Moreover, the production decision can be taken separately from the decision on the forward position, since the implicit characterization of $Q^\star$ is independent of h. In a sense, just the opposite is true if forwards are subject to default, as stated in the following proposition.

Proposition 2: *If forward contracts are subject to counter-party default risk, then (i) the marginal cost of production at the optimal output quantity $Q^\star$ is higher than the forward price if it is optimal to buy forward contracts, i.e., $h^\star < 0 \Rightarrow c'(Q^\star) > F_0$, (ii) the marginal cost of production at the optimal output quantity $Q^\star$ is lower than the forward price if it is optimal to sell forward contracts, i.e., $h^\star > 0 \Rightarrow c'(Q^\star) < F_0$, (iii) separation of the production decision from the decision on the forward contracts is not possible.*

Proposition 2 states that no firm that holds non-zero forward positions should have marginal production costs (at the optimal output level) equal to the forward price. Instead of a point of attraction, as is the case with default-free forwards, the forward price seems to be a point of repulsion. A formal proof of this result can be found in the appendix, but an intuitive reasoning is as follows: For the moment, assume that forward contracts are default-free. If marginal costs were below the forward price, increasing output by a marginal unit and selling a marginal unit in the forward market would generate an extra profit that is risk free, i.e., the firm would leave a free lunch on the table if it did not do so. An analogous argument applies to the case of marginal production costs above the forward price. Reducing output by a marginal unit and buying the marginal unit in the forward market generates a free lunch. Now assume that forwards are default risky. If marginal production costs equal the forward price and long positions in forwards are held, the firm can increase expected utility by simultaneously increasing output and reducing the forward position by a marginal unit. Such a transaction essentially leaves the firm´s profit unchanged if forwards do not default or are out of the money, but increases the profit if forwards are in the money and default occurs. Thus, if the firm would choose an output that leads to marginal costs equal to the forward price, it would essentially forego a free lottery. A similar argument applies to the case with short positions in forwards and marginal production costs equal to the forward price.

If forwards contracts are default risky, the forward price is not the only price variable that influences the choice of the optimal output. Since forwards provide no protection against price risk in the case of default, it is intuitive that output generally depends on the distribution of the spot price at time 1. Moreover, output will generally depend on the particular form of the firm´s utility function and it would no longer be possible to separate the production decision from the hedging decision.

Proposition 2 relates output to the forward price, but does not provide a comparison between the output levels $Q^\star$ that result for default-free and default-risky forward contracts. Similarly, proposition 1 refers to the forward position in proportion to output, but not to the absolute number $h^\star$ of forward contracts. The following proposition 3 sheds some light on the effects of default risk on $Q^\star$ and $h^\star$.

Proposition 3: *If default-free and default-risky forward contracts on the price $\tilde{P}_1$ have zero expected profits, forward contracts subject to counter-party default risk lead to a lower optimal output level $Q^\star$ than default-free forward contracts.*

Proposition 3 compares a situation where the firm has access to default-free forwards with a situation where only default-risky forwards are available. Both default-free and default-risky contracts have an expected profit of zero from the firm´s perspective. Under this assumption, the firm sells forward contracts according to proposition 1 and the forward price of default-free contracts is lower than the forward price of default-risky contracts. The assumption of zero expected profits allows us to identify a pure risk effect on $Q^\star$. According to proposition 3, this risk effect leads to a lower optimal output quantity if forwards are subject to default. Intuitively, since forward contracts can no longer achieve a perfect hedge of the price risk, a risk averse firm reduces its price exposure by reducing output. With respect to $h^\star$, propositions 3 and 1 imply that the absolute number of forward contracts sold is lower if forward contracts are default-risky, even though the hedge ratio is not affected. Thus, there is a production effect on the forward position, but not on the hedge ratio.

3 Model Extensions

If forward contracts earn zero expected profits, the hedge ratio is the same for default-free and default-risky forward contracts. Does this robustness result still hold if the firm follows more sophisticated risk management strategies that consider the hedging of default risk or the impact of other sources of risk, like basis risk or recovery rate risk? The following subsections develop some extensions of the basic model to analyze this question.

3.1 Credit Derivatives on Forwards

Counter-party default risk can in principle be managed by means of derivatives contracts whose payments are contingent on the counterparty´s default.[5] However, how should such a risk management strategy be designed? More specifically, if a credit derivative that is written on the default-risky forward contract is available, how should a firm change its usage of forward contracts and how many credit derivatives should it take?

To answer these questions, consider a first model extension. In addition to forwards, the firm can now enter into a credit derivative contract. The credit contract costs a premium equal to K_0 that is payable at time 0, and promises to cover any losses that arise from the default of a sold forward contract at time 1, i.e., $\tilde{I}\max\left[(F_0 - \tilde{P}_1), 0\right]$. However, a credit derivative can itself be default-risky. Thus, it is assumed that the actual payment of the credit contract at time 1 equals $(1-\tilde{J})\tilde{I}\max\left[(F_0 - \tilde{P}_1), 0\right]$, where $\tilde{J}$ denotes a Bernoulli(q) distributed random variable that indicates whether the credit derivative does default ($J = 1$) or does not default ($J = 0$). All three random variables $\tilde{P}_1$, $\tilde{I}$, and $\tilde{J}$ can generally be stochastically dependent. With the additional investment opportunity in credit derivatives, the firm´s profit becomes:

$$\begin{aligned}\tilde{\Pi} &= \tilde{P}_1 Q - c(Q) + h(F_0 - \tilde{P}_1) - \tilde{I}\max\left[h(F_0 - \tilde{P}_1), 0\right] \\ &\quad + z\left[\tilde{I}(1-\tilde{J})\max\left[(F_0 - \tilde{P}_1), 0\right] - K_0\,(1+r)\right], \end{aligned} \tag{3}$$

[5]Markets for different types of such credit derivatives have expanded rapidly over the last few years. See e.g. Triennial Central Bank Survey (2002).

where z denotes the number of credit derivatives bought and r is the firm´s risk-free borrowing or lending rate for the period from time 0 to time 1.

The firm has to decide simultaneously about the output, the forward position, and the position in the credit derivative, i.e., it solves the following maximization problem:

$$\max_{Q,h,z} E[U(\tilde{\Pi})], \quad \text{s.t. } \tilde{\Pi} \text{ as defined in equation (3).} \tag{4}$$

The first result for the extended model, stated in proposition 4, refers to the hedge ratio $h^\star/Q^\star$.

Proposition 4: *If credit derivatives on forward contracts are available, the optimal hedge ratio $h^\star/Q^\star$ is equal to one if and only if the expected profit from selling forward contracts is zero, i.e.,* $F_0 = E[\tilde{P}_1] + E\left[\tilde{I}\,\max[(F_0 - \tilde{P}_1), 0]\right] \Leftrightarrow h^\star/Q^\star = 1.$

According to proposition 4, the robustness result of proposition 1 still holds. If forward contracts earn zero expected profits, credit risk does not affect the hedge ratio $h^\star/Q^\star$, even in an extended setting with credit derivatives. In essence, this result is due to the fact that a hedge ratio of one still leads to constant profits if forwards do not default or the firm´s forward position is out of the money. The only difference between a situation with and without credit derivatives lies in the resulting profit level, which equals $F_0Q^\star - c(Q^\star) - z^\star K_0(1+r)$ in the former case and $F_0Q^\star - c(Q^\star)$ in the latter. If forwards default and the firm´s forward position is in the money, the number of forward contracts h has no impact on profits, irrespective of $z^\star$. Thus, whatever the firm´s optimal position in credit derivatives, the optimal hedge ratio will be $h^\star/Q^\star = 1$.

Another interesting issue that can be addressed by means of the extended model is the usage of credit derivatives. The next proposition refers to this issue, in particular to the ratio $z^\star/h^\star$.

Proposition 5 : *(i) If expected profits of forward contracts are zero and credit derivatives are default free, the ratio* $z^\star/h^\star$ *is equal to one if and only if expected profits of credit derivatives are zero, i.e.,* $K_0(1+r) = E\left[\tilde{I}\,\max[(F_0 - \tilde{P}_1), 0]\right] \Leftrightarrow z^\star/h^\star = 1$.
(ii) If expected profits of forward contracts are zero and credit derivatives are default risky, the ratio $z^\star/h^\star$ *is greater than zero and smaller than one if expected profits of the credit derivative are zero, i.e.,* $K_0(1+r) = E\left[\tilde{I}\,(1-\tilde{J})\,\max[(F_0 - \tilde{P}_1), 0]\right] \Rightarrow 0 < z^\star/h^\star < 1$.

The first part of proposition 5 considers a reference case with default-free credit derivatives. If default-free credit derivatives are available, we are essentially back in a world with default-free forward contracts. A derivatives portfolio that combines a short position of one unit of default-risky forwards with a long position of one unit of default-free credit contracts earns a profit equal to $F_0 - K_0(1+r) - \tilde{P}_1$, the same profit that a short position in a default-free forward contract with forward price $F_0 - K_0(1+r)$ would earn. Since it is possible to eliminate price risk, a full hedge is optimal if expected profits from derivatives contracts are zero. The availability of default-free credit derivatives also implies that the optimal production quantity does not depend on the distribution of $\tilde{P}_1$ and the specific form of the utility function. It is easy to show that $c'(Q^\star) = F_0 - K_0(1+r)$ must hold.

The second part of proposition 5 considers the more realistic case of default-risky credit derivatives. A first message is that every risk averse firm should buy a positive amount of credit derivatives if expected profits from these contracts are zero. The availability of credit derivatives is utility increasing for every risk averse firm. A generally positive demand $z^\star$ for credit contracts might also be seen as an explanation for the development of markets for such instruments.

A second message is that firms should not try to fully hedge credit risk with credit contracts, i.e., a firm should choose a ratio $z^\star/h^\star$ smaller than one. Since credit derivatives are subject to default, risk can not be totally avoided, but only transferred by means of different instruments. In a sense, the second part of proposition 5 states that it is optimal to diversify credit risk. Consider the extreme cases $z/h = 0$ and $z/h = 1$. In the first case, only forward contracts with forward price F_0 and default probability $E(\tilde{I})$ are held. In the second case, only "synthetic" forward contracts with forward price $F_0 - K_0(1+r)$ and default probability $E(\tilde{I}\tilde{J})$ are held.

However, both strategies are not optimal. An optimal derivatives position consists of a mixture of the two "types" of forward contracts. Since the default risks of the two "types" of contracts are not perfectly correlated [6], i.e., $Cov(\tilde{I}, \tilde{I}\tilde{J}) < 1$, such a kind of diversification is intuitively reasonable.

3.2 Basis Risk

Basis risk is an important issue if price risk is managed with standardized futures contracts, because futures often do not perfectly fit a firm´s price exposure. However, even if forward contracts are used, which can in principle be tailored to the customers´ needs, basis risk might still be relevant. If a firm´s price exposure refers to the price of a very specialized product, there might be no counterparty that offers an appropriate contract under acceptable terms, and the firm is forced to enter into a cross hedge with other forwards. Another potential reason for basis risk of forward hedges is that firms do not know the exact timing of their revenues a priori.

Benninga, Eldor and Zilcha (1983, 1984) derived an important robustness result with regard to an optimal hedging strategy with default-free forwards under additive basis risk. They assume that at time 1 the forward price F_1 need not be equal to the price P_1. Instead, from the perspective of time 0, the following relation between $\tilde{P}_1$ and $\tilde{F}_1$ holds:

$$\tilde{P}_1 = a + b\,\tilde{F}_1 + \tilde{\epsilon}, \tag{5}$$

where a and b are constant parameters and $\tilde{\epsilon}$ is a zero-mean random variable that is independent of $\tilde{F}_1$. No further assumptions are made on the distributions of $\tilde{F}_1$ and $\tilde{\epsilon}$. Benninga, Eldor, and Zilcha´s (1983, 1984) result states that if forward contracts earn an expected profit of zero, the optimal forward position $h^\star$ equals $b\,Q$ for every risk averse firm. Thus, the optimal hedging strategy is robust with respect to the specific form of the utility function and the price distribution. In particular, the variance minimizing hedge $h^\star = b\,Q$ does not rely on very specific

[6] A correlation of one occurs only in the uninteresting case of a useless credit derivative that generally defaults if the forward contract defaults.

assumptions like a quadratic utility function or an exponential utility function and normally distributed profits.[7]

The presence of both basis risk and counter-party default risk raises two related questions. First, if and how does the robustness result of proposition 1 generalize to the case with additional basis risk? Second, does the robustness result by Benninga, Eldor, and Zilcha (1983, 1984) still hold if forward contracts are subject to default? These questions are addressed by means of a second model extension.

Assume that forwards are default risky and have an additive basis risk, as given in equation (5). Further assume that $\tilde{\epsilon}$ is stochastically independent from both $\tilde{F}_1$ and $\tilde{I}$. Then, the firm´s profit becomes:

$$\tilde{\Pi} = \tilde{F}_1 b\, Q + (a + \tilde{\epsilon})Q - c(Q) + h(F_0 - \tilde{F}_1) - \tilde{I}\max\left[h(F_0 - \tilde{F}_1), 0\right]. \qquad (6)$$

Based on the profit equation (6), the firm chooses the forward position $h^\star$ and the output $Q^\star$ that maximize the expected utility of profits. The following proposition characterizes the optimal hedge ratio $h^\star/Q^\star$.

Proposition 6 : *If there exists an additive basis risk, a variance minimizing hedging strategy is optimal if and only if forward contracts earn zero expected profits from the firm´s perspective, i.e.,* $F_0 = E[\tilde{F}_1] + E\left[\tilde{I}\ \max[(F_0 - \tilde{F}_1), 0]\right] \Leftrightarrow h^\star/Q^\star = b$, *if* $b > 0$, *and* $F_0 = E[\tilde{F}_1] - E\left[\tilde{I}\ \max[(\tilde{F}_1 - F_0), 0]\right] \Leftrightarrow h^\star/Q^\star = b$, *if* $b < 0$.

Proposition 6 states that Benninga, Eldor, and Zilcha´s (1983, 1984) robustness result with respect to the optimal forward position is still valid even with default-risky forwards. If a firm expects that forwards earn zero profits on average, it should choose a hedge ratio that minimizes the variance of profits. Starting from the robustness result of proposition 1, one can conclude that the introduction of an additional basis risk has the same effect on the hedge ratio for default-free and default-risky forwards. In this sense, proposition 1 also applies to the case with additional basis risk.

[7]However, note that Benninga, Eldor, and Zilcha´s (1983, 1984) robustness result does not hold for other forms of basis risk, as was shown by Briys, Crouhy, and Schlesinger (1993) and Adam-Müller (2003).

3.3 Recovery Rate Risk

A firm´s profit uncertainty due to a possible default of forward contracts has generally two sources. The first one is the uncertainty on whether a default will occur. The second one is the uncertainty on the amount that is lost in the case of a default. In the model variants analyzed so far, the "loss given default" depends on a single risk factor, the price $\tilde{P}_1$. However, a more realistic approach would also consider another risk factor, a stochastic recovery rate. This subsection provides a third model extension that explores the impact of recovery rate risk on the hedge ratio $h^\star/Q^\star$.

Consider the setting of the basic model, but assume that in the case of default only a fraction $(1-\tilde{R})$ of the gains from forward contracts is lost. The random variable $\tilde{R}$ denotes the recovery rate of a forward contract and can take values between zero and one. It need not be stochastically independent of $\tilde{P}_1$ and $\tilde{I}$. With a stochastic recovery rate of defaulted forward contracts, the firm´s profit becomes:

$$\tilde{\Pi} = \tilde{P}_1 Q - c(Q) + h(F_0 - \tilde{P}_1) - \tilde{I}(1-\tilde{R}) \max\left[h(F_0 - \tilde{P}_1), 0\right]. \tag{7}$$

Note that the model by Holthausen (1979) results as a special case for $R \equiv 1$ and the basic model of Section 2 results as a special case for $R \equiv 0$. In the following, a positive probability that $\tilde{R}$ is strictly smaller than one and greater than zero is assumed. Proposition 7 characterizes the optimal hedge ratio $h^\star/Q^\star$ for the case of zero expected profits.

Proposition 7: *If forward contracts have a stochastic recovery rate $\tilde{R}$ and the expected profit from selling forward contracts is zero, the hedge ratio $h^\star/Q^\star$ is greater than one, i.e., $F_0 = E[\tilde{P}_1] + E\left[\tilde{I}(1-\tilde{R})\max[(F_0 - \tilde{P}_1), 0]\right] \Rightarrow h^\star/Q^\star > 1$.*

In the model variants of the previous subsections, default risk had no effect on the hedge ratio. According to proposition 7, a firm´s reaction to default risk is to sell even more forward contracts, i.e., take a larger forward position in absolute terms. At first sight, this result might be astonishing. Should we not expect that a firm reduces its use of an instrument that makes the firm´s profit sensitive to additional sources of risk?

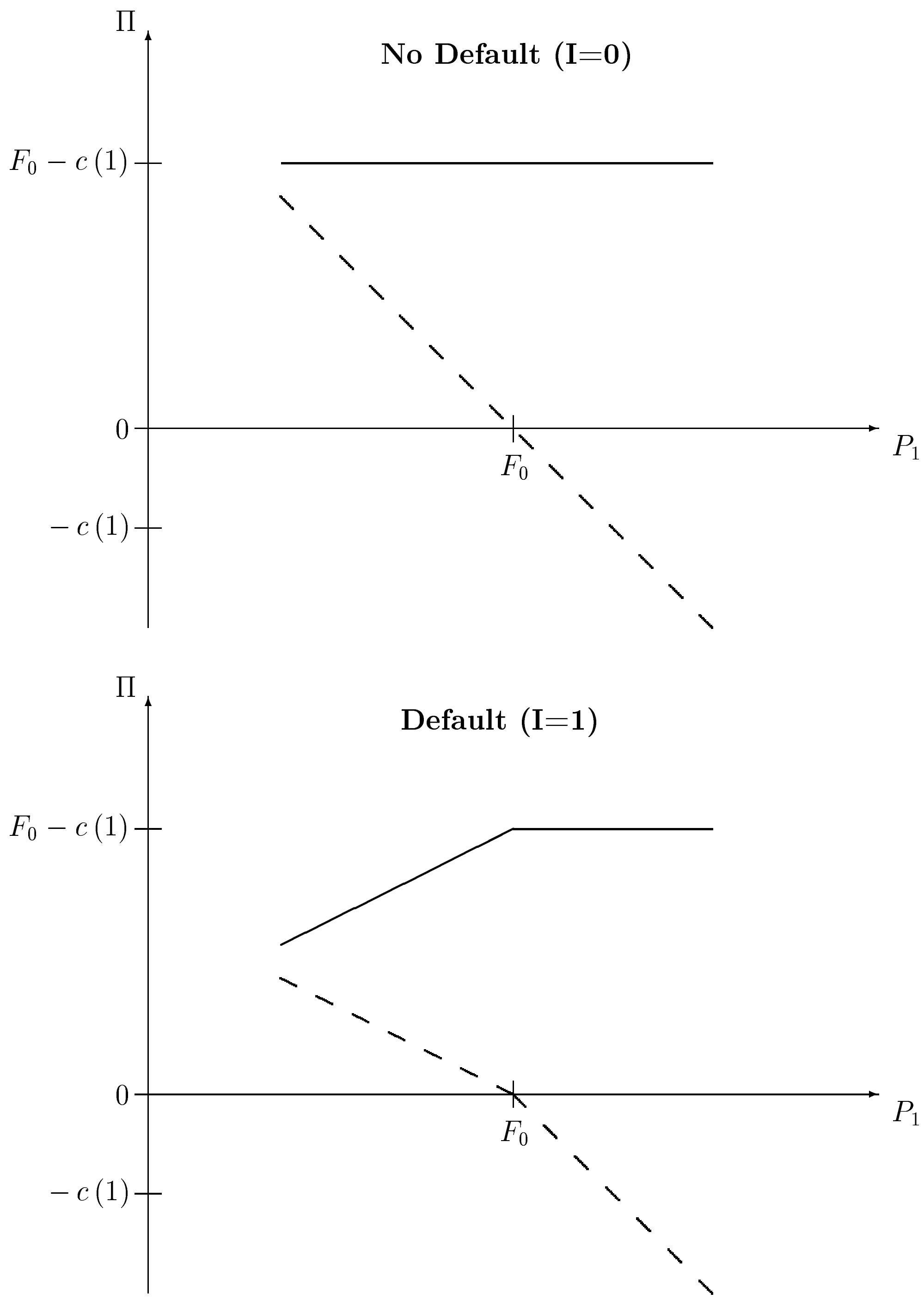

Figure 2: Firm´s profit and payoff of a sold forward contract with recovery rate $R = 0.5$ as functions of P_1

The figure shows the firm´s profit (solid lines) and the payoff of a sold forward contract with recovery rate $R = 0.5$ (dashes lines) as functions of the price P_1. The upper part refers to the case when the forward does not default, the lower part refers to the case when the forward defaults. The figure assumes that $Q^\star = 1$ and $h/Q^\star = 1$.

Figure 2 provides a view into the benefits of overhedging, i.e., the choice of a hedge ratio greater than one, if the recovery rate lies in the interval $(0,1)$. Similar to figure 1, figure 2 depicts the firm´s profit Π and the payoff of a sold forward contract as a function of the price P_1. Again, it is assumed that $Q^\star = 1$ and $h/Q^\star = 1$. However, in contrast to figure 1, there is a recovery rate of $R = 0.5$. With a full hedge, constant profits of $F_0 - c(1)$ are obtained if the forward does not default or $P_1 > F_0$. However, if the forward defaults and $P_1 \leq F_0$, the profit strictly increases with P_1 and the payoff of the forward contract strictly decreases with P_1. The sale of a further marginal unit of forward contracts leads to higher profits in states where they are most beneficial, i.e., where profits are low and marginal utility is high. Thus, the fact that some gains from forward contracts can be realized even in the case of default offers an opportunity to shift some profits to the low profit states by means of overhedging.

In practice, the choice of a hedge ratio greater than one might be difficult to defend, in particular if the reasoning behind it relies on the uncertain gains from defaulted forward contracts. In any case, all model variants considered in this paper suggest that default risk of forward contracts is no valid reason to underhedge, i.e., to choose a hedge ratio below one. However, with respect to the absolute number of forward contracts sold, one must also consider the production effect of default risk.

4 Summary and Conclusions

Many non-financial firms that use OTC derivatives are confronted with the problem of counter-party default risk. Therefore, how firms should consider default risk of derivatives contracts in their risk management strategies is an important question. This paper makes a first step towards an answer. Within a model of a risk-averse competitive firm under price uncertainty, it derives several fundamental results on a firm´s optimal forward position and output quantity if forward contracts are subject to default.

A first set of results identifies the conditions under which default risk does not matter. The basic result states that the hedge ratio is not affected by default risk if forward contracts earn zero expected profits. In this case, full hedging is generally optimal. A model extension shows that an important robustness result by Benninga, Eldor, and Zilcha (1983, 1984) still holds for default-risky contracts. If there is an additive basis risk and the expected profits of forwards are zero, a variance minimizing hedge ratio is optimal for general concave utility functions and general price distributions, irrespective of the default risk. Thus, variance minimizing hedging strategies, that are attractive with respect to tractability and implementation, can be theoretically justified under rather general assumptions.

The analysis given in this paper leads to a clear recommendation: Firms should not reduce hedge ratios in response to default risk. On the contrary, an extended model with a stochastic recovery rate suggests that one should rather increase hedge ratios. The literature has identified several valid reasons for underhedging, like quantity risk (Benninga, Eldor, and Zilcha (1985) and Adam-Müller (1997)), specific types of basis risk (Briys, Crouhy, and Schlesinger (1993) and Adam-Müller (2003)), liquidity risk (Korn (2003)), and a comparative advantage in risk-taking (Stulz(1996)). The presence of default risk is not one of them.

The paper also shows in which respect default risk does matter. Even if expected profits of forwards are zero, the number of forward contracts sold will generally be reduced by default risk if output is endogenous. The reason is that a firm should produce less if it has to rely on default-risky forward contracts instead of default-free ones. If the expected profits of forward contracts are non-zero, default risk might also affect the speculative component of a firm´s forward position. Moreover, with default-risky forwards, it is no longer possible to determine the optimal output quantity independently from the optimal forward position. The forward price is no longer the only price information that affects production.

If forward contracts are subject to default, credit derivatives written on forwards should be used to some extent. In a sense, a firm should diversify between the default risk of the forward and the default risk of the credit derivative. This result of the paper suggests that it might also be valuable to diversify between different counterparties in the forward market. Another open question is whether a firm should stick to forwards at all, or use derivatives with non-linear payoffs instead if default risk is taken into account. These issues are left for future research.

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Appendix

Proof of Proposition 1:

The proposition is proved in two steps. In the first step, I show that the stated equivalence holds under the assumption that $h^\star \geq 0$. The second step completes the proof under the assumption that $h^\star < 0$.

Step 1: Consider the first derivative of the firm´s objective function $E[U(\tilde{\Pi})]$ with respect to h, evaluated at $h = Q^\star$:

$$E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right].$$

The firm´s objective function is strictly concave in h for non-negative values of h under the stated assumption that $U'' < 0$. Therefore, under the additional assumption that $h^\star \geq 0$, the following equivalence holds:

$$E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right] \gtreqless 0 \quad (8)$$

$$\Leftrightarrow \quad h^\star / Q^\star \gtreqless 1. \quad (9)$$

The expectation on the left hand side of inequality (8) can be rewritten as follows:

$$\begin{aligned}
& E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right] \quad (10)\\
= \; & E\left[E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right] | \, \tilde{I}\right]\\
= \; & p\,E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star, \, I = 1\right]\\
& +(1-p)\,E\left[U'(\tilde{\Pi})(F_0 - \tilde{P}_1) | \, h = Q^\star, \, I = 0\right].
\end{aligned}$$

If $h = Q^\star$ and $I = 0$ (no default), the profit Π equals $F_0 Q^\star - c(Q^\star)$, i.e., it is not stochastic. Thus, the marginal utility $U'(\Pi | \, h = Q^\star, \, I = 0)$ is a constant function of P_1. In the case of default ($h = Q^\star, I = 1$), the same constant function results for $P_1 \geq F_0$. For $P_1 < F_0$, the marginal utility decreases with increasing P_1, but $(F_0 - P_1) - \max[(F_0 - P_1), 0]$ is always equal to zero. Therefore, the marginal utility $U'(\Pi) | \, h = Q^\star, \, I = 0)$ can be written in front of the expectations in the last two lines of equation (10):

$$\begin{aligned}
& p\,E\left[U'(\tilde{\Pi})\left[(F_0-\tilde{P}_1)-\max[(F_0-\tilde{P}_1),0]\right]|\,h=Q^\star,\,I=1\right] && (11)\\
& +(1-p)\,E\left[U'(\tilde{\Pi})(F_0-\tilde{P}_1)|\,h=Q^\star,\,I=0\right] &&\\
= \;& p\,U'(\Pi|\,h=Q^\star,\,I=0)\,E\left[(F_0-\tilde{P}_1)-\max[(F_0-\tilde{P}_1),0]|\,I=1\right] && (12)\\
& +(1-p)\,U'(\Pi|\,h=Q^\star,\,I=0)\,E\left[(F_0-\tilde{P}_1)|\,I=0\right]. &&
\end{aligned}$$

Since $U'(\Pi|\,h=Q^\star,\,I=0)$ is positive, the following equivalence holds:

$$\begin{aligned}
& p\,U'(\Pi|\,h=Q^\star,\,I=0)\,E\left[(F_0-\tilde{P}_1)-\max[(F_0-\tilde{P}_1),0]|\,I=1\right] &&\\
& \quad +(1-p)\,U'(\Pi|\,h=Q^\star,\,I=0)\,E\left[(F_0-\tilde{P}_1)|\,I=0\right] \gtreqless 0 &&\\
\Leftrightarrow \;& p\,E\left[(F_0-\tilde{P}_1)-\max[(F_0-\tilde{P}_1),0]|\,I=1\right] && (13)\\
& \quad +(1-p)\,E\left[(F_0-\tilde{P}_1)|\,I=0\right] \gtreqless 0. &&
\end{aligned}$$

The terms on the left hand side of inequality (13) can be written in more compact form as

$$\begin{aligned}
& p\,E\left[(F_0-\tilde{P}_1)-\max[(F_0-\tilde{P}_1),0]|\,I=1\right] &&\\
& +(1-p)\,E\left[(F_0-\tilde{P}_1)|\,I=0\right] &&\\
= \;& E\left[E\left[(F_0-\tilde{P}_1)-\tilde{I}\max[(F_0-\tilde{P}_1),0]\right]|\,\tilde{I}\right] &&\\
= \;& E\left[(F_0-\tilde{P}_1)-\tilde{I}\,\max[(F_0-\tilde{P}_1),0]\right]. && (14)
\end{aligned}$$

Using expression (14), the following equivalence has been shown:

$$\begin{aligned}
& E\left[U'(\tilde{\Pi})\left[(F_0-\tilde{P}_1)-\tilde{I}\max[(F_0-\tilde{P}_1),0]\right]|\,h=Q^\star\right] \gtreqless 0 && (15)\\
\Leftrightarrow \;& E\left[(F_0-\tilde{P}_1)-\tilde{I}\,\max[(F_0-\tilde{P}_1),0]\right] \gtreqless 0. && (16)
\end{aligned}$$

The desired result follows from the equivalence of equations (8) and (9) and the equivalence of equations (15) and (16), i.e.,

$$\begin{aligned}
F_0 & \gtreqless E[\tilde{P}_1]+E\left[\tilde{I}\,\max[(F_0-\tilde{P}_1),0]\right]\\
\Leftrightarrow h^\star/Q^\star & \gtreqless 1.
\end{aligned}$$

Step 2: Now assume that $h^\star < 0$. To complete the proof of the proposition, it remains to show that $F_0 < E[\tilde{P}_1] + E\left[\tilde{I}\ \max[(F_0 - \tilde{P}_1), 0]\right]$ must hold in this case.

For a negative value of h, the profit in equation (1) becomes

$$\tilde{\Pi} = \tilde{P}_1 Q - c(Q) + h(F_0 - \tilde{P}_1) - h\,\tilde{I}\min\left[(F_0 - \tilde{P}_1), 0\right]. \tag{17}$$

Since $h^\star$ is assumed to be negative, it must satisfy the following necessary condition:

$$E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}\min[(F_0 - \tilde{P}_1), 0]\right]\right] = 0. \tag{18}$$

The left hand side of equation (18) can be rewritten as follows:

$$\begin{aligned}
& E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}\min[(F_0 - \tilde{P}_1), 0]\right]\right] \\
= \; & E\left[U'(\tilde{\Pi})(F_0 - \tilde{P}_1)\right] - E\left[U'(\tilde{\Pi})\tilde{I}\min[(F_0 - \tilde{P}_1), 0]\right] \\
= \; & E\left[U'(\tilde{\Pi})\right]E(F_0 - \tilde{P}_1) - Cov(U'(\tilde{\Pi}), \tilde{P}_1) - E\left[U'(\tilde{\Pi})\tilde{I}\min[(F_0 - \tilde{P}_1), 0]\right]
\end{aligned} \tag{19}$$

The last expectation on the right hand side of equation (19) is negative, provided that forwards can default and do not offer an arbitrage opportunity. The covariance $Cov(U'(\tilde{\Pi}), \tilde{P}_1)$ is also negative. To see this, note that for negative h the profit, as given in equation (17), strictly increases with P_1, irrespective of a default or a non-default of the forward contract. Since marginal utility strictly decreases with profits, the covariance term must be negative.

We can now conclude that for equation (18) to hold, $E\left[U'(\tilde{\Pi})\right]E(F_0 - \tilde{P}_1)$ must be negative. Since marginal utility is always positive, this condition implies that F_0 must be smaller than $E(\tilde{P}_1)$. Therefore, it has been shown that $F_0 < E[\tilde{P}_1] + E\left[\tilde{I}\ \max[(F_0 - \tilde{P}_1), 0]\right]$ is necessary to give a firm the incentive to hold a long positions in forwards. □

Proof of Proposition 2:

The first order condition with respect to the output quantity Q reads:

$$\begin{aligned} &E\left[U'(\tilde{\Pi})(\tilde{P}_1 - c'(Q))\right] = 0 \\ \Leftrightarrow\quad &c'(Q) = \frac{E\left[U'(\tilde{\Pi})\tilde{P}_1\right]}{E\left[U'(\tilde{\Pi})\right]}. \end{aligned} \tag{20}$$

If $h^\star$ is greater than zero, the following first order condition holds:

$$\begin{aligned} &E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}max[F_0 - \tilde{P}_1, 0]\right]\right] = 0 \\ \Leftrightarrow\quad &F_0 - \frac{E\left[U'(\tilde{\Pi})\,\tilde{I}\,max[F_0 - \tilde{P}_1, 0]\right]}{E[U'(\tilde{\Pi})]} = \frac{E\left[U'(\tilde{\Pi})\tilde{P}_1\right]}{E\left[U'(\tilde{\Pi})\right]}. \end{aligned} \tag{21}$$

Since the two expressions on the right hand sides of equations (20) and (21) are equal, the following expression for the marginal costs results:

$$c'(Q) = F_0 - \frac{E\left[U'(\tilde{\Pi})\,\tilde{I}\,max[F_0 - \tilde{P}_1, 0]\right]}{E[U'(\tilde{\Pi})]}. \tag{22}$$

The random variable $U'(\tilde{\Pi})\,\tilde{I}\,max[F_0 - \tilde{P}_1, 0]$ takes a positive value at least for one realization of $\tilde{P}_1$, as long as the forward contract can default and does not provide an arbitrage opportunity. However, $U'(\tilde{\Pi})\,\tilde{I}\,max[F_0 - \tilde{P}_1, 0]$ can never be negative. Therefore, the expectation of $U'(\tilde{\Pi})\,\tilde{I}\,max[F_0 - \tilde{P}_1, 0]$ must be positive and $c'(Q) < F_0$ holds, which proofs the first part of the proposition.

The proof of the second part is analogous. If $h^\star < 0$, the following first order condition holds:

$$\begin{aligned} &E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}min[F_0 - \tilde{P}_1, 0]\right]\right] = 0 \\ \Leftrightarrow\quad &F_0 - \frac{E\left[U'(\tilde{\Pi})\,\tilde{I}\,min[F_0 - \tilde{P}_1, 0]\right]}{E[U'(\tilde{\Pi})]} = \frac{E\left[U'(\tilde{\Pi})\tilde{P}_1\right]}{E\left[U'(\tilde{\Pi})\right]}. \end{aligned} \tag{23}$$

Equations (20) and (23) imply that

$$c'(Q) = F_0 - \frac{E\left[U'(\tilde{\Pi})\,\tilde{I}\,min[F_0 - \tilde{P}_1, 0]\right]}{E[U'(\tilde{\Pi})]}. \tag{24}$$

Since $U'(\tilde{\Pi})\,\tilde{I}\,min[F_0 - \tilde{P}_1, 0]$ is a non-positive random variable that is negative at least in one state, provided the forward contract can default and does not provide an arbitrage opportunity, its expectation is negative. Thus, is follows from equation (24) that $c'(Q) > F_0$.

Finally, equations (22) and (24) show that the optimal production quantity depends on the specific form of the utility function, the distribution of the profit $\tilde{\Pi}$, and the distribution of the price $\tilde{P}_1$. Because the distribution of $\tilde{\Pi}$ depends on the forward position, it is not possible to take the production decision independently from the hedging decision. Separation does no longer hold if forward contracts are default risky. □

Proof of Proposition 3:

Consider the first order condition of the maximization problem (2) with respect to Q:

$$\begin{aligned} & E\left[U'(\tilde{\Pi})(\tilde{P}_1 - c'(Q))\right] = 0 \\ \Leftrightarrow\quad & c'(Q) = \frac{E\left[U'(\tilde{\Pi})\tilde{P}_1\right]}{E\left[U'(\tilde{\Pi})\right]} \\ \Leftrightarrow\quad & c'(Q) = E[\tilde{P}_1] + \frac{Cov\left[U'(\tilde{\Pi})\,,\,\tilde{P}_1\right]}{E\left[U'(\tilde{\Pi})\right]}. \end{aligned}$$

According to proposition 1, the optimal hedge ratio $h^\star/Q^\star$ equals one if forward contracts have zero expected profits, irrespective of the default probability of the forward. If forward contracts are default-free and the firm follows an optimal hedging policy, the profit Π is a constant function of the price P_1. In this case, the covariance $Cov\left[U'(\tilde{\Pi})\,,\,\tilde{P}_1\right]$ is zero. If forward contracts are default risky and the firm follows an optimal hedging policy, the profit is a constant function of P_1 if $I = 0$ or $P_1 \geq F_0$, and a strictly increasing function of P_1 if $I = 1$ and $P_1 < F_0$. Since $U'(\Pi)$ is a strictly decreasing function of Π, the covariance $Cov\left[U'(\tilde{\Pi})\,,\,\tilde{P}_1\right]$ must be negative. Such a negative covariance implies that the marginal production costs are lower than in the case of default-free forwards. Due to the convex cost function, this result implies that the optimal output quantity $Q^\star$ must also be lower. □

Proof of Proposition 4:

The proof is completely analogous to the proof of proposition 1. However, some comments on the essential step of the proof, the equality of expressions (11) and (12), might be useful. In the more general model with credit derivatives, if $h = Q^\star$ and $I = 0$ (no default), the profit Π equals $F_0 Q^\star - c(Q^\star) - z^\star K_0(1+r)$, i.e., the profit is still nonstochastic and the marginal utility $U'(\Pi|\, h = Q^\star,\, I = 0)$ is a constant function of P_1. In the case of default $(h = Q^\star, I = 1)$, the same constant function results for $P_1 \geq F_0$. Since $(F_0 - P_1) - \max[(F_0 - P_1), 0]$ is always equal to zero for $P_1 < F_0$, the marginal utility $U'(\Pi|\, h = Q^\star,\, I = 0)$ can still be written in front of the expectations in the last two lines of equation (10). Expressions (11) and (12) are equal even in the more general model with credit derivatives. □

Proof of Proposition 5:

First, consider the second part of the proposition. It follows from proposition 4 that $h^\star = Q^\star > 0$ under the assumption of zero expected profits of forward contracts. Thus, the ratio $z^\star/h^\star$ is always defined. Since the firm´s objective function is strictly concave in z over the whole real line, the ratio $z^\star/h^\star$ is smaller than one if the first derivative of $E[U'(\tilde{\Pi})]$ with respect to z is negative for $z = h^\star$. Therefore, we have to show that the following inequality holds:

$$E\left[U'(\tilde{\Pi})\left[\tilde{I}(1-\tilde{J})\max[(F_0 - \tilde{P}_1), 0] - K_0(1+r)\right] |\, z = h^\star\right] < 0. \qquad (25)$$

The left hand side of inequality (25) can be written as

$$\begin{aligned}
& E\left[U'(\tilde{\Pi})\left[\tilde{I}(1-\tilde{J})\max[(F_0 - \tilde{P}_1), 0] - K_0(1+r)\right] |\, z = h^\star\right] \\
= \;& E\left[E\left[U'(\tilde{\Pi})\left[\tilde{I}(1-\tilde{J})\max[(F_0 - \tilde{P}_1), 0] - K_0(1+r)\right] |\, z = h^\star\right] |\, \tilde{I}, \tilde{J}\right] \\
= \;& -p_{I=1,J=1} E\left[U'(\tilde{\Pi}) K_0(1+r)|\, z = h^\star, I = 1, J = 1\right] \qquad (26) \\
& + p_{I=1,J=0} E\left[U'(\tilde{\Pi})\max[(F_0 - \tilde{P}_1), 0] - K_0(1+r)|\, z = h^\star, I = 1, J = 0\right] \\
& - p_{I=0,J=1} E\left[U'(\tilde{\Pi}) K_0(1+r)|\, z = h^\star, I = 0, J = 1\right] \\
& - p_{I=0,J=0} E\left[U'(\tilde{\Pi}) K_0(1+r)|\, z = h^\star, I = 0, J = 0\right],
\end{aligned}$$

where $p_{I=1,J=1}, p_{I=1,J=0}, p_{I=0,J=1}$, and $p_{I=0,J=0}$ denote the unconditional probabilities for the corresponding events.

Division of the right hand side of equation (26) by $U'(\Pi|z = h^\star, I = 0) = U'(\Pi|z = h^\star, I = 1, J = 0)$, which is a non-random scalar, leads to the following expression:

$$
\begin{aligned}
& -p_{I=1,J=1} E\left[(U'(\tilde{\Pi})/U'(\Pi|z = h^\star, I = 0)) K_0(1+r) |\, z = h^\star, I = 1, J = 1 \right] \\
& +p_{I=1,J=0} E\left[\max[(F_0 - \tilde{P}_1), 0] - K_0(1+r) |\, z = h^\star, I = 1, J = 0 \right] \qquad (27) \\
& -p_{I=0,J=1} E\left[K_0(1+r) |\, z = h^\star, I = 0, J = 1 \right] \qquad (28) \\
& -p_{I=0,J=0} E\left[K_0(1+r) |\, z = h^\star, I = 0, J = 0 \right] \qquad (29)
\end{aligned}
$$

Since the marginal utility is generally lowest if the forward contract does not default $(I = 0)$, the following inequality holds:

$$
\begin{aligned}
& -p_{I=1,J=1} E\left[(U'(\tilde{\Pi})/U'(\Pi|z = h^\star, I = 0))\, K_0(1+r) |\, z = h^\star, I = 1, J = 1 \right] \\
< \quad & -p_{I=1,J=1} E\left[K_0(1+r) |\, z = h^\star,\ I = 1,\ J = 1 \right] \qquad (30)
\end{aligned}
$$

The expressions (27), (28), (29), and the expression on the right hand side of inequality (30) sum up to the following unconditional expectation:

$$
E\left[\tilde{I}(1 - \tilde{J}) \max[(F_0 - \tilde{P}_1), 0] - K_0(1+r) \right]. \qquad (31)
$$

Under the assumption that credit derivatives earn zero expected profits, the above expectation (31) is zero. Therefore, it has been shown that the following inequality holds:

$$
\frac{E\left[U'(\tilde{\Pi}) \left[\tilde{I}(1 - \tilde{J}) \max[(F_0 - \tilde{P}_1), 0] - K_0(1+r) \right] |\, z = h^\star \right]}{U'(\Pi|\, z = h^\star,\ I = 0)} < 0.
$$

Since marginal utility is positive, this result implies that inequality (25) also holds.

The first part of the proposition also follows from the above argument. One has just to consider that in the case with default-free forward contracts inequality (30) becomes an equality. Therefore inequality (25) also holds as an equality, i.e., the necessary condition for an optimal solution, which is also a sufficient condition under our assumptions, is fulfilled.

It remains to show that $z^\star/h^\star$ is strictly greater than zero if credit derivatives are subject to default. Based on the first order condition, a strictly positive position in credit derivatives is optimal if the following inequality holds:

$$E\left[U'(\tilde{\Pi})\left[\tilde{I}(1-\tilde{J})\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\right]\mid z=0\right]>0. \tag{32}$$

The left hand side of equation (32) can be rewritten as follows:

$$\begin{aligned}
& E\left[U'(\tilde{\Pi})\left[\tilde{I}(1-\tilde{J})\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\right]\mid z=0\right] \\
= \; & E\left[E\left[U'(\tilde{\Pi})\left[\tilde{I}(1-\tilde{J})\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\right]\mid z=0\right]\mid \tilde{I},\tilde{J}\right] \\
= \; & -p_{I=1,J=1}E\left[U'(\tilde{\Pi})K_0(1+r)\mid z=0,I=1,J=1\right] \\
& +p_{I=1,J=0}E\left[U'(\tilde{\Pi})(\max[(F_0-\tilde{P}_1),0]-K_0(1+r))\mid z=0,I=1,J=0\right] \\
& -p_{I=0,J=1}E\left[U'(\tilde{\Pi})K_0(1+r)\mid z=0,I=0,J=1\right] \\
& -p_{I=0,J=0}E\left[U'(\tilde{\Pi})K_0(1+r)\mid z=0,I=0,J=0\right] \\
= \; & -p_{I=1,J=1}E\left[U'(\tilde{\Pi}\mid z=0,I=1,J=1)\right]K_0(1+r) \\
& +p_{I=1,J=0}E\left[U'(\tilde{\Pi}\mid z=0,I=1,J=0)\right]E\left[\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\mid I=1,J=0\right] \\
& +p_{I=1,J=0}Cov\left[U'(\tilde{\Pi}),\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\mid z=0,\ I=1,\ J=0\right] \\
& -p_{I=0,J=1}E\left[U'(\tilde{\Pi}\mid z=0,I=0,J=1)\right]K_0(1+r) \\
& -p_{I=0,J=0}E\left[U'(\tilde{\Pi}\mid z=0,I=0,J=0)\right]K_0(1+r) \\
= \; & E\left[U'(\tilde{\Pi}\mid z=0)\right]E\left[\tilde{I}(1-\tilde{J})\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\right] \\
& +p_{I=1,J=0}Cov\left[U'(\tilde{\Pi}),\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\mid z=0,\ I=1,\ J=0\right] \\
= \; & p_{I=1,J=0}Cov\left[U'(\tilde{\Pi}),\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\mid z=0,\ I=1,\ J=0\right],
\end{aligned} \tag{33}$$

where the last equality follows from the assumption that credit derivatives earn zero expected profits, i.e., $E\left[\tilde{I}(1-\tilde{J})\max[(F_0-\tilde{P}_1),0]-K_0(1+r)\right]=0$.

Thus, inequality (32) holds if the covariance term on the right hand side of equation (33) is positive. The positive sign of the conditional covariance can be seen as follows: If the forward defaults and no credit derivatives are held, the firm´s profit is an increasing function of P_1, and strictly increasing for $P_1<F_0$. The payoff of a long position in credit derivatives is decreasing with P_1, and strictly decreasing for $P_1<F_0$. Since the marginal utility is itself a strictly decreasing function of the firm´s profit, it must be positively correlated with the payoff of the credit derivative. □

Proof of Proposition 6:

The proof follows the same line of argument as the proof of proposition 1. In the first step, it is assumed that $h^\star \geq 0$. In the second step, it is assumed that $h^\star < 0$.

Step 1: Assume that b is greater than zero and consider the first derivative of the firm´s objective function $E[U(\tilde{\Pi})]$ with respect to h, evaluated at $h = bQ^\star$. Because of the strict concavity of the optimization problem for non-negative values of h, and because of the additional assumption that $h^\star \geq 0$, it is a necessary and sufficient condition for $h^\star = bQ^\star$ that this derivative equals zero, i.e.,

$$E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\max[(F_0 - \tilde{F}_1), 0]\right] | \, h = bQ^\star\right] = 0. \tag{34}$$

The left hand side of equation (34) can be rewritten as follows:

$$\begin{aligned} & E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\max[(F_0 - \tilde{F}_1), 0]\right] | \, h = bQ^\star\right] \\ = \; & E\left[E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\max[(F_0 - \tilde{F}_1), 0]\right] | \, h = bQ^\star\right] |\tilde{I}, \tilde{\epsilon}\right] \\ = \; & E\left[p\,E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \max[(F_0 - \tilde{F}_1), 0]\right] | \, h = bQ^\star, \, I = 1\right]\right. \\ & \left. + \, (1-p)\,E\left[U'(\tilde{\Pi})(F_0 - \tilde{F}_1)| \, h = bQ^\star, \, I = 0\right] |\tilde{\epsilon}\right]. \end{aligned} \tag{35}$$

Conditional on $\tilde{\epsilon}$, one can apply the same reasoning as is the proof of proposition 1 and write $U'(\tilde{\Pi}| \, h = bQ^\star, \, I = 0, \, \tilde{\epsilon})$ in front of the inner expectations in the last two lines of equation (35):

$$\begin{aligned} & E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\max[(F_0 - \tilde{F}_1), 0]\right] | \, h = bQ^\star\right] \\ = \; & E\left[U'(\tilde{\Pi}| \, h = bQ^\star, \, I = 0, \, \tilde{\epsilon})\,p\,E\left[(F_0 - \tilde{F}_1) - \max[(F_0 - \tilde{F}_1), 0]| \, I = 1, \, \tilde{\epsilon}\right]\right. \\ & \left. + \, U'(\tilde{\Pi}| \, h = bQ^\star, \, I = 0, \, \tilde{\epsilon})\,(1-p)\,E\left[(F_0 - \tilde{F}_1)| \, I = 0, \, \tilde{\epsilon}\right]\right] \\ = \; & E\left[U'(\tilde{\Pi}| \, h = bQ^\star, \, I = 0, \, \tilde{\epsilon})E\left[(F_0 - \tilde{F}_1) - \tilde{I}\,\max[(F_0 - \tilde{F}_1), 0]| \, \tilde{\epsilon}\right]\right] \end{aligned}$$

Since $\tilde{\epsilon}$ is stochastically independent of both $\tilde{I}$ and $\tilde{F}_1$ by assumption, we obtain the following result:

$$\begin{aligned} & E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\max[(F_0 - \tilde{F}_1), 0]\right] | \, h = bQ^\star\right] \\ = \; & E\left[U'(\tilde{\Pi}| \, h = bQ^\star, \, I = 0, \, \tilde{\epsilon})\right] E\left[(F_0 - \tilde{F}_1) - \tilde{I}\,\max[(F_0 - \tilde{F}_1), 0]\right] \end{aligned}$$

If forwards earn zero expected profits, i.e., $E\left[(F_0 - \tilde{F}_1) - \tilde{I}\,\max[(F_0 - \tilde{F}_1), 0]\right] = 0$, equation (34). An analogous argument can be made for the case $b < 0$.

Step 2: It remains to show that the assumption $h^\star < 0$ leads to a contradiction. For a negative value of h, the profit in equation (6) becomes

$$\tilde{\Pi} = \tilde{F}_1 bQ + (a+\epsilon)Q - c(Q) + h(F_0 - \tilde{F}_1) - h\,\tilde{I}\min\left[(F_0 - \tilde{F}_1), 0\right]. \qquad (36)$$

Since $h^\star$ is assumed to be negative, it must satisfy the following necessary condition:

$$E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\min[(F_0 - \tilde{F}_1), 0]\right]\right] = 0. \qquad (37)$$

The left hand side of equation (37) can be rewritten as follows:

$$\begin{aligned} & E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{F}_1) - \tilde{I}\min[(F_0 - \tilde{F}_1), 0]\right]\right] \\ = \; & E\left[U'(\tilde{\Pi})(F_0 - \tilde{F}_1)\right] - E\left[U'(\tilde{\Pi})\tilde{I}\min[(F_0 - \tilde{F}_1), 0]\right] \qquad (38) \\ = \; & E\left[U'(\tilde{\Pi})\right] E(F_0 - \tilde{F}_1) - Cov(U'(\tilde{\Pi}), \tilde{F}_1) - E\left[U'(\tilde{\Pi})\tilde{I}\min[(F_0 - \tilde{F}_1), 0]\right] \end{aligned}$$

The last expectation on the right hand side of equation (38) is negative, provided that forwards can default and do not offer an arbitrage opportunity. The covariance $Cov(U'(\tilde{\Pi}), \tilde{F}_1)$ is also negative. To see this, note that for negative h the profit, as given in equation (36), strictly increases with F_1, irrespective of a default or a non-default of the forward contract and irrespective of the realisation of the basis risk $\tilde{\epsilon}$. Since $\tilde{F}_1$ and $\tilde{\epsilon}$ are stochastically independent and marginal utility strictly decreases with profits, the covariance term must be negative.

We can now conclude that for equation (37) to hold, $E\left[U'(\tilde{\Pi})\right] E(F_0 - \tilde{F}_1)$ must be negative. Because marginal utility is always positive, this condition implies that F_0 must be smaller than $E(\tilde{F}_1)$. Since it has been assumed in the proposition that $F_0 = E[\tilde{F}_1] + E\left[\tilde{I}\ \max[(\tilde{F}_1 - F_0), 0]\right]$, this finding is contrary to the optimality of a long position in forwards.

An analogous contradiction results for the case $b < 0$. $\square$

Proof of Proposition 7:

It can be shown by the same arguments used in the second step of the proof of proposition 6 that a negative value of $h^\star$ leads to a contradiction. Thus, we assume that $h^\star \geq 0$ and concentrate on a region where the objective function is strictly concave in h.

The optimal hedge ratio $h^\star/Q^\star$ is greater than one if the first derivative of the objective function with respect to h, evaluated at $h = Q^\star$, is greater than zero, i.e.,

$$E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}(1-\tilde{R})\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right] > 0. \qquad (39)$$

To show that inequality (39) holds, rewrite the the left hand side of the inequality as follows:

$$\begin{aligned}
& E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}(1-\tilde{R})\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right] \\
= \; & E\left[E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}(1-\tilde{R})\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star\right] | \, \tilde{I}\right] \\
= \; & p\, E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - (1-\tilde{R})\max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star, \, I = 1\right] \\
& +(1-p)\, E\left[U'(\tilde{\Pi})(F_0 - \tilde{P}_1) | \, h = Q^\star, \, I = 0\right] \\
= \; & p\, E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \max[(F_0 - \tilde{P}_1), 0]\right] | \, h = Q^\star, \, I = 1\right] \qquad (40) \\
& +p\, E\left[U'(\tilde{\Pi})\tilde{R}\max[(F_0 - \tilde{P}_1), 0] | \, h = Q^\star, \, I = 1\right] \\
& +(1-p)\, E\left[U'(\tilde{\Pi})(F_0 - \tilde{P}_1) | \, h = Q^\star, \, I = 0\right].
\end{aligned}$$

Now divide the right hand side of equation (40) by $U'(\Pi | \, h = Q^\star, \, I = 0)$, which is a non-random scalar. As a consequence, the marginal utility cancels from the first and third expectations. With respect to the first expectation, the same reasoning as in the proof of proposition 1 applies. After division, the sum on the right hand side of equation (40) becomes

$$\begin{aligned}
& p\, E\left[(F_0 - \tilde{P}_1) - \max[(F_0 - \tilde{P}_1), 0] | \, I = 1\right] \qquad (41) \\
& +p\, E\left[(U'(\tilde{\Pi})/U'(\Pi | \, h = Q^\star, \, I = 0))\tilde{R}\max[(F_0 - \tilde{P}_1), 0] | \, h = Q^\star, \, I = 1\right] \\
& +(1-p)\, E\left[(F_0 - \tilde{P}_1) | \, I = 0\right].
\end{aligned}$$

For the next step, note that the following inequality (42) holds:

$$E\left[(U'(\tilde{\Pi})/U'(\Pi|\, h = Q^\star,\, I = 0))\tilde{R}\max[(F_0 - \tilde{P}_1), 0]|\, h = Q^\star,\, I = 1\right] \;>\; E\left[\tilde{R}\max[(F_0 - \tilde{P}_1), 0]|\, h = Q^\star,\, I = 1\right] \qquad (42)$$

The reason is that the random variable $U'(\tilde{\Pi}|\, h = Q^\star,\, I = I)/U'(\Pi|\, h = Q^\star,\, I = 0)$ takes values that are strictly greater than one if $P_1 < F_0$.

Inequality (42) implies that

$$\begin{aligned} & p\,E\left[(F_0 - \tilde{P}_1) - \max[(F_0 - \tilde{P}_1), 0]|\, I = 1\right] \\ & +p\,E\left[(U'(\tilde{\Pi})/U'(\Pi|\, h = Q^\star,\, I = 0))\tilde{R}\max[(F_0 - \tilde{P}_1), 0]|\, h = Q^\star,\, I = 1\right] \\ & +(1-p)\,E\left[(F_0 - \tilde{P}_1)|\, I = 0\right] \;> \\ & p\,E\left[(F_0 - \tilde{P}_1) - \max[(F_0 - \tilde{P}_1), 0]|\, I = 1\right] \\ & +p\,E\left[\tilde{R}\max[(F_0 - \tilde{P}_1), 0]|\, I = 1\right] \\ & +(1-p)\,E\left[(F_0 - \tilde{P}_1)|\, I = 0\right] \;= \\ & E\left[(F_0 - \tilde{P}_1) - \tilde{I}\,(1 - \tilde{R})\,\max[(F_0 - \tilde{P}_1), 0]\right] \end{aligned}$$

Since expected profits of short positions in forwards are assumed to be zero, i.e., $E\left[(F_0 - \tilde{P}_1) - \tilde{I}\,(1 - \tilde{R})\,\max[(F_0 - \tilde{P}_1), 0]\right] = 0$, it has been shown that the following inequality (43) holds:

$$\frac{E\left[U'(\tilde{\Pi})\left[(F_0 - \tilde{P}_1) - \tilde{I}(1 - \tilde{R})\max[(F_0 - \tilde{P}_1), 0]\right] |\, h = Q^\star\right]}{U'(\Pi|\, h = Q^\star,\, I = 0)} > 0. \qquad (43)$$

Because marginal utility is positive, inequality (43) implies that inequality (39) also holds. □