On partial defaults in portfolio credit risk

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Abstract

Most credit portfolio models calculate the loss distribution of a portfolio consisting solely of performing counterparts. However, conservative default definitions cause considerable insecurity about the loss of a non-performing counterpart for a long time after its default. We develop two models that account for defaulted counterparts in the calculation of the economic capital. First, we model the portfolio of non-performing counterparts stand-alone. The second approach derives the integrated loss distribution for the non-performing and the performing portfolio. Both calculations are supplemented by formulae for contributions of the single counterpart to the economic capital. Calibrating the models allows for an impact study and a comparison with Basel II.

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1 Introduction

Regulators currently have made an effort to align capital requirements with actual credit risk. Banks develop internal models to take account of their specific portfolio structure. Foreseeing the portfolio loss, even if only known by its probability distribution, is of central interest. The risk contributions of the individual exposure derive from it. On the level of the single event, the individual loss given default (LGD) identifies that part of the exposure - usually valued at the time of default (EAD) - that cannot be regained in the course of settling the claims[1]. Many models assume the net exposure, i.e. the LGD and the exposure , to be known in advance. However, the definition of bankruptcy in banking is conservative in order to put an early incentive towards intensive work-out of endangered investments. A bank might well be exposed to a counterpart years after the default definition is fulfilled, especially for private debt portfolios where post-default trading of the debt is rare (Carey (1998)). Financial intermediates are typically two to four years exposed after the last cash paid (Schuermann (2005)), an expectation of one and a half years was found for large bank loans by Gupton et al. (2000). As a consequence, one observation in banking is that the losses vary materially from their expectations (Gupton et al. (2000))[2]. If LGD's did vary independently, diversification arguments suggest that only the expected LGD enters the calculations (Tasche (2004)). Basel II supports the dependence assumption for the LGD's across counterparts by demanding the measurement of the LGD to be larger than the expected LGD. An economic

[1]We define the EAD as unsecured portion of the outstanding amount, net of risk mitigation, but gross of collateralization.

[2]See also Calem and LaCour-Little (2004) for an argument of randomness in the loss given foreclosure for mortgage loans.

explanation is the dependence of the LGD on the collateral, an asset. The value of many - if not all - assets is related to the general economic activity leading to dependent values of collateral and hence dependent LGD's.

We develop two methods, one for the non-performing portfolio exclusively, the "stand-alone method", and a second that integrates the performing portfolio, the "integrated method". In the first, the LGD depends on the general economic activity and we express the dependence by a one-factor model, independently of Tasche (2004). Distributional assumptions are of minor interest in this simple model, we illustrate the idea with an LGD depending on a normal economic activity and a normal idiosyncratic risk. The economic capital and capital requirements, i.e. risk contributions for the individual debtors are derived.

Usually, a portfolio owner wants to know the economic capital for the entire portfolio as calculations for separate portfolios always lead to an overestimation on the aggregate level. We base the second method on a mixture model, a common tool for modeling dependent events in finance (see e.g. McNeil et al. (2005); Credit Suisse First Boston (CSFB) (1997)). The recent work of Bürgisser et al. (2001) integrates a random LGD in the calculation of the portfolio loss distribution. Section 3 briefly reviews the method. On basis of that model we combine the performing and the non-performing portfolio.

For both models, we calculate the loss distribution and its variance and decompose the economic capital into contributions for the single exposures.

The calibration of the models based on a default study is described. Additionally, an impact study is presented to compare the derived methods with regulatory capital requirements.

2 One-factor model

The default of a counterpart is economically fixed to the date of the first default on an allying payment. It is common in financial institutions to make provisions on the event of default. In the model with deterministic LGD's no further insecurity is left, and thus the counterpart must be excluded from the calculation of the future-loss distribution. However, the definition of default implies that the magnitude of final loss is not yet known. The final overall loss may be greater than the provision. The LGD needs a stochastic model.

We want to calculate the loss distribution for the portfolio of defaulted counterparts whose exposure is not yet completely depreciated. Sparse data on LGD forces a sparse model, hence we will assume normality for all random variables in this section. However, correlations between LGD's of different counterparts in one year - empirically found e.g. by Gupton et al. (2000) - must be modelled in order not to underestimate a portfolio's credit risk. We restrict the model to a lump-sum correlation of ρ and one additional parameter for the volatility.

We base our model on the residual exposure of a defaulted counterpart A, i.e. net of provisions and write-off's, in the portfolio of all defaults $\mathcal{E}$ at the beginning of the risk horizon, say a year. To lay out the methodological details, we denote the change of provision for counterpart A during the year by Δ_A, whereas δ_A denotes the change of the provision relative to the EAD e_A, Δ_A/e_A. The EAD is used for two reasons: First of all, to avoid dominance of small exposures (leading potentially to unreasonably large relative changes in the residual exposure) in the calibration and second to enable comparison with Basel II regulations as well as our second model.

The relative changes are assumed to follow the same probability distribution for all $A \in \mathcal{E}$. Furthermore, we assume the relative changes to depend on a latent factor Y modeling the economic activity, made responsible for the deviation of the mean from 0:

$$\delta_A = Y + \epsilon_A \quad A \in \mathcal{E}. \tag{1}$$

The idiosyncratic variability of the relative changes for counterpart A in the year is represented by the noise ϵ_A. We assume the Y and the ϵ_A's to follow independent normal distributions $N(0, \sigma_Y^2)$ and $N(0, \sigma_\epsilon^2)$. The variance σ_δ^2 of δ_A is $\sigma_Y^2 + \sigma_\epsilon^2$. The relation of σ_Y^2 and σ_ϵ^2 determines the common correlation $\rho = corr(\delta_A, \delta_{\tilde{A}}) = \sigma_Y^2/(\sigma_Y^2 + \sigma_\epsilon^2)$.

We have now sufficient information to calculate the credit Value-at-Risk of the non-performing portfolio $\mathcal{E}$. The loss generated by the portfolio until the end of the year is $L_n = \sum_{A\in\mathcal{E}} e_A \delta_A$. The variance is $Var(L_n) = \left(\sum_{A\in\mathcal{E}} e_A^2 + \sum_{A,\tilde{A}\in\mathcal{E},A\neq\tilde{A}} e_A e_{\tilde{A}} \rho\right) \sigma_\delta^2$.

Denote by $e = \sum_{A\in\mathcal{E}} e_A$ the total exposure of the non-performing portfolio and use as concentration measure the Herfindahl-Hirschmann index $H := (\sum_{A\in\mathcal{E}} e_A^2)/e^2$ (Hirschmann (1964)). With the valid approximation $\sum_{A,\tilde{A}\in\mathcal{E},A\neq\tilde{A}} e_A e_{\tilde{A}}/e^2 \approx 1$, the loss variance is

$$Var(L_n) = e^2(H + \rho)\sigma_\delta^2.$$

In the limiting case of an infinitesimal granular portfolio, H is 0 and the variance reduces to the systematic effect of Y namely $e^2\sigma_Y^2$ and is positive if $\rho > 0$. The economic capital at level γ (as well as the credit Value-at-Risk because of the expectation 0) is

$$EC_{n,\gamma} = e u_\gamma (H + \rho)^{1/2} \sigma_\delta, \tag{2}$$

where u_γ denotes the γ-quantile of the standard normal distribution. Typical values for γ are $99.95, 99.5, 99, 90$ and 75%, their u_γ's are 3.29, 2.58, 2.33, 1.28 and 0.68.

The risk contributions for the separate exposures can now be attributed, e.g. proportionally with respect to the exposure[3]

$$ec_A := \frac{e_A EC_{n,\gamma}}{e} = e_A u_\gamma (H+\rho)^{1/2}\sigma_\delta. \tag{3}$$

The calibration of the model, i.e. the estimation of ρ and σ_δ, is postponed to Section 4. An impact study - quantifying also the portfolio-specific measure H - together with a comparison to regulatory capital treats Section 5. In the meantime we present a model that takes the diversification potential with the performing portfolio into account and avoids the normality assumptions.

If the economic capital for the non-performing portfolio and for the performing portfolio are calculated separately, the overall economic capital is overstated. Both credit Value-at-Risks need to be added. The resulting level of confidence for a default is now in general higher than the nominal levels aimed at. To illustrate that, assume both losses L_n of the non-performing portfolio and L_p of the performing portfolio to be independent normal random variables with variances σ_n^2 and σ_p^2. The ratio of the sum of the separate credit Value-at-Risks and the credit Value-at-Risk for the combined portfolio is $(\sigma_n+\sigma_p)(\sigma_n^2+\sigma_p^2)^{-1/2}$ and larger than 1 due to the triangular inequality.

[3]A conceptionally more advanced idea is to use the derivative of the loss variance with respect to the single exposure (Credit Suisse First Boston (CSFB) (1997)). However, it can be seen that the model (1) implies the derivative of the loss variance to be linear in the exposure (for an infinitesimally granular portfolio). Hence, an exposure-linear attribution is appealing.

The relation is independent of the level γ. In this particulary easy example, using the inverse ratio as factor rectifies the conservativeness.

3 Integrating the performing portfolio

Modeling more carefully, we must strive to calculate the economic capital for the entire portfolio in order to account for the diversification potentials. Additional to the non-performing $\mathcal{E}$, consider a performing portfolio $\mathcal{A}$. The default of a counterpart A can simply be seen as Bernoulli variable $I_A \sim B(\mu_A)$, and thus the loss deriving from the performing portfolio is

$$L_p = \sum_{A \in \mathcal{A}} \nu_A I_A. \tag{4}$$

The net exposure ν_A is the product of the (forecast for the) exposure e_A and the LGD λ_A. The key difference between δ_A in the previous model in Section 2 and λ_A here is that δ_A "traces" the loss history via the provisioning process whereas λ_A records only the overall LGD. An additional difference is that, instead of individual LGD's, we model "portfolio LGD's" by omitting the individual LGD noise (ϵ). Diversification suggests the impact of the individual noise on the the economic capital to be negligible. Consider the following simple one-factor model for the LGD (cf. Bürgisser et al. (2001)):

$$\lambda_A = l_A \Lambda, \tag{5}$$

where Λ is a random variable with expectation 1 and variance σ_Λ^2, that is independent of the defaults $I_A,\ A \in \mathcal{A}$[4]. The independence of default and

[4]The one-factor model (5) in this section assumes that the relative LGD's of different counterparts are perfectly correlated. One may relieve the assumption of one latent LGD factor Λ and allow for inhomogeneous LGD correlations. The calculations are similar to those given here and omitted for the sake of brevity.

LGD is doubtable. In a large study Altman et al. (2002) prove positive correlations as well as Frye (2000); Hu and Perraudin (2002). However, they admit that a meta analysis by Carey and Gordy (2001) finds negligible correlations unless restricting the study period. We like to add that even an indication of negative correlation may be found in Carey (1998). Despite of the stronger evidence for positive correlation, we like to argue that the technical feasibility of the independence assumption outperforms the lack of reality. The same seems to be true for all commercial models we know, as they as well assume independence.

In order to account for (stochastic) dependencies between defaults of different counterparts, we assume a mixture model. The probability of default depends on a latent (random) economic activity factor X with $E(X) = 1$ and $Var(X) = \sigma_X^2$:

$$\mu_A = p_A X. \tag{6}$$

The models seems to be restrictive, however, Bürgisser et al. (1999) shows how to reduce a model with several correlated economic activity factors (e.g. for several industries) to this form. As usual, we assume the defaults I_A to be independent, conditional on X.

The portfolio loss can now be written as

$$\tilde{L}_p := \sum_{A \in \mathcal{A}} e_A l_A \Lambda I_A = \Lambda \sum_{A \in \mathcal{A}} e_A l_A I_A = \Lambda L_p. \tag{7}$$

Clearly, the expected loss $E(\tilde{L}_p)$ is again equal to $E(L_p)$.

The loss distribution can be calculated as

$$\begin{aligned}
\tilde{F}_p(k) &:= P(\tilde{L}_p \le k) = P(\Lambda \le k/L_p), \quad k/0 := \infty \\
&= \sum_{n\ge 0} P(\Lambda \le k/L_p \mid L_p = n)\; P(L_p = n) \\
&= P(L_p = 0) + \sum_{n\ge 1} P(\Lambda \le k/n)\; P(L_p = n) \\
&= f_{L_p}(0) + \sum_{n\ge 1} f_{L_p}(n) F_\Lambda(k/n), \qquad (8)
\end{aligned}$$

with $f_{L_p}(n) := P(L_p = n)$ and $F_\Lambda(n) := P(\Lambda \le n)$, $n = 0, 1, \ldots$. The distribution of the performing portfolio $f_{L_p}(n)$ can be calculated using any portfolio model. The distribution of the LGD needs to be chosen, we will argue for a generalized Beta-distribution in the following Section 4.

To the loss of the performing portfolio (7) we add now the loss of the non-performing portfolio. We simply model the defaults as Bernoulli experiments with parameter 1.

The loss for a given time horizon of the portfolio $\mathcal{A} \cup \mathcal{E}$ is

$$\tilde{L} := \sum_{A\in\mathcal{A}} e_A \lambda_A \; I_A + \sum_{A\in\mathcal{E}} e_A \lambda_A = \tilde{L}_p + L_n \qquad (9)$$

where $\tilde{L}_p$ is defined in (7) and $L_n := \sum_{A\in\mathcal{E}} e_A\lambda_A$ represents the loss from the non-performing portfolio $\mathcal{E}$.

Owing to model (5) ($\lambda_A = l_A\Lambda$) we can decompose $\tilde{L}_p$ and L_n into $\tilde{L}_p = \Lambda L_p$ and $L_n = \Lambda\eta$ with $L_p := \sum_{A\in\mathcal{A}} e_A l_A \; I_A$ and the deterministic expected net exposure $\eta := \sum_{A\in\mathcal{E}} e_A \; l_A$, respectively. For the ease of notation let $\tilde{L} := \Lambda L$ with $L := L_p + \eta$.

Note that the definitions $\tilde{L}_p$ and L_p fit the definitions in the model without defaulted counterparts (see (7)). L_n, the (random) loss arising from the sub-portfolio of defaulted counterparts, in contrast to the definition in

Section 2, integrates the initial provision into the loss[5].

The calculation of the loss distribution is analogous to the distribution of $\tilde{L}$ above: $\tilde{F}(k) = P(\tilde{L} \leq k) = f_L(0) + \sum_{n\geq 1} f_L(n)\, F_\Lambda(k/n)$, where now $f_L(n)$ only depends on $f_{L_p}(n)$ because $f_L(n) = P(L = n) = P(L_p = n-\eta) = f_{L_p}(n-\eta)$. The leading term $f_L(0)$ can only be positive if $\mathcal{E} = \emptyset$, the case is already covered by the expression (8) and hence not considered now. As a first result we now have a procedure for calculating the portfolio credit risk. The credit Value-at-Risk at level γ for the loss $\tilde{L}$ of the combined portfolio (see formula (9)) is given by

$$CreditVaR_\gamma = \inf\left\{ k : \sum_{n\geq 1} f_{L_p}(n-\eta)\, F_\Lambda(k/n) > \gamma \right\}. \tag{10}$$

Under the assumptions for formula (10) the economic capital for a joint portfolio of not defaulted and defaulted counterparts is given by $EC_\gamma = CreditVaR_\gamma - (\sum_{A\in\mathcal{A}} p_A e_A l_A + \sum_{A\in\mathcal{E}} e_A l_A)$, because $E(\tilde{L}) = E(\Lambda)(E(L_p) + \eta) = \sum_{A\in\mathcal{A}} p_A e_A l_A + \sum_{A\in\mathcal{E}} e_A l_A$.

An important issue in portfolio risk is the attribution of the risk to the responsible counterparts. A standard procedure is to consider the portfolio loss variance $Var(\tilde{L})$ as risk measure and attribute the risk according to the change in variance as the net exposure ν_A changes (Credit Suisse First

[5] The point of the initial provision (wright-off) is a important difference in the two models: In the stand-alone model the focus on provision changes leads to an economic capital irrespective of the expected LGD. Whereas in the integrated model the economic risk and includes the expected LGD and hence needs to be subtracted in arrear. Both models are feasible if the necessary data to calibrate the different LGD models are available, which is often the case.

Boston (CSFB) (1997)). The loss variance here calculates as

$$\begin{aligned} Var(\tilde{L}) &= E(Var(\tilde{L} \mid \Lambda)) + Var(E(\tilde{L} \mid \Lambda)) \\ &= E(\Lambda^2 Var(L_p + \eta)) + Var(\Lambda E(L)) \\ &= E(\Lambda^2) Var(L_p) + E(L)^2 Var(\Lambda) \\ &= (1 + \sigma_\Lambda^2) Var(L_p) + \sigma_\Lambda^2 (E(L_p) + \eta)^2. \end{aligned}$$

With the notation of the PD model (6), a short calculation yields

$$Var(L_p) = \sum_{A \in \mathcal{A}} e_A^2 l_A^2 \, p_A \left(1 - p_A \left(1 + \sigma_X^2\right)\right) + \sigma_X^2 E(L_p)^2, \tag{11}$$

with the expected loss of the performing portfolio being $E(L_p) = \sum_{A \in \mathcal{A}} p_A e_A l_A$.

An additive risk attribution is guaranteed for the definition $\tilde{vc}_A := (e_A/2)(\partial Var(\tilde{L})/\partial e_A)$. The variance contribution is now twofold, according to whether counterpart A defaulted or not:

$$\tilde{vc}_A = \begin{cases} p_A e_A l_A (d_A + (E(L_p) + \eta)\sigma_\Lambda^2) & ; A \in \mathcal{A} \\ e_A l_A (E(L_p) + \eta)\sigma_\Lambda^2 & ; A \in \mathcal{E} \end{cases} \tag{12}$$

with $d_A := (1 + \sigma_\Lambda^2)(e_A l_A (1 - p_A) + \sigma_X^2 (E(L_p) - p_A l_A e_A))$. The representation for the performing portfolio includes a penalty for large single EAD, e_A, reflected by the quadratic component[6]. Contrary, for the defaulted counterparts no single exposure concentration penalty is necessary, a simple percentage of the EAD (equivalent to the "capital requirement (K)" in the notation of Basel II) is accurate[7]. As pointed out in the beginning of the section, diversification between the performing and the non-performing

[6]The representation (12) is simular to the variance contribution for deterministic exposure (see Credit Suisse First Boston (CSFB) (1997)).

[7]This is an analogy to Basel II (Basel Commitee on Banking Supervision (2004), paragraph 272), where the linearity arises from the assumption of an infinitely granular portfolio.

portfolio is possible and the interaction between the portfolio becomes manifest in the joint parameters. The expected loss of the performing portfolio $E(L_p)$ influences the variance contributions for the non-performing portfolio and vice versa with η. The LGD parameter σ_Λ^2 affects both.

However, the contribution to the variance is only an intermediate step. A key question in finance is the allocation of economic capital for pricing, costing and budgeting. We need a portion of the economic capital attributable to each counterpart so that the contributions add up to the economic capital. We do already have a notion of cause and effect for the dependence of the loss variance on the exposure of each counterpart. The economic capital and the loss variance are closely related, e.g. for the assumption of a normally distributed loss, the EC is a linear transformation of the loss volatility. EC and loss variance are measures for the potential deviation of the loss from its expectation. As usual, we will now assume that the economic capital exhibits the same sensitivity with respect to the exposure of each counterpart as the variance does. The number

$$ec_A = \frac{\tilde{vc}_A}{\sum_B \tilde{vc}_B}\, EC_\gamma = \frac{\tilde{vc}_A}{Var(\tilde{L})}\, EC_\gamma \tag{13}$$

constitutes an approximate contribution of the exposure of counterpart A to the economic capital obeying $\sum_A ec_A = EC_\gamma$.[8]

[8] Two alternatives exist that establish a cause-effect attribution of the economic capital: First, one may calculate the economic capital with the whole portfolio and again with the portfolio leaving out one counterpart. The difference between the two values can be interpreted as the risk contribution of the counterpart. However, the approach has two disadvantages. On the one hand, the attribution is not additive, and the sum of all contributions thus derived is usually less than the economic capital. The "late coming counterpart" profits from the existing diversification. The procedure could be refined by using an idea from game theory. One could add the counterparts subsequently to the

The incorporation of the non-performing counterparts into the total loss calculation affects the risk contributions for performing counterparts. The changes are noticeable, and thus one aim may be to separate risk contributions for non-performing counterparts from the performing portfolio. To this end, we can calculate the economic capital for the portfolio without the defaulted counterparts and derive risk contributions for that. In a second stage we need to calculate the distribution (and EC) for the entire portfolio to determine the increase in EC caused by the non-performing portfolio.

Starting with the performing portfolio $\mathcal{A}$ in analogy to the described procedure, we define for $A \in \mathcal{A}$

$$ec_A := \frac{\tilde{vc}_A^{(1)}}{\sum_{B\in\mathcal{A}} \tilde{vc}_B^{(1)}} EC_\gamma^{(1)} \tag{14}$$

where $EC_\gamma^{(1)} := CreditVaR_\gamma^{(1)} - E(\tilde{L}_p)$ and $CreditVaR_\gamma^{(1)} = \inf\{k : P(\tilde{L}_p \leq k) > \gamma\}$ denotes the Value-at-Risk. The variance contributions for the performing exposure is denoted by $\tilde{vc}_A$. The contribution to the loss variance of an exposure in the performing portfolio (including stochastic LGD) is

$$\begin{aligned} \tilde{vc}_A^{(1)} &:= \frac{e_A}{2}\frac{\partial Var(\tilde{L}_p)}{\partial e_A} = \frac{e_A}{2}\left((1+\sigma_\Lambda^2)\frac{\partial Var(L_p)}{\partial e_A} + \sigma_\Lambda^2 \frac{\partial E(L_p)^2}{\partial e_A}\right) \\ &= p_A e_A l_A \Big((1+\sigma_\Lambda^2)\Big(e_A l_A (1-p_A) \\ &+ \sigma_X^2 (E(L_p) - p_A l_A e_A)\Big) + E(L_p)\sigma_\Lambda^2\Big). \end{aligned}$$

The ec_A is now calculated as in (14). For $A \in \mathcal{E}$ the marginal contribution of the non performing portfolio $EC_\gamma - EC_\gamma^{(1)}$ can be distributed according

portfolio and average over all possible sequences. Unfortunately, the computational effort for large portfolios is sizeable even for the leave-one-out approach (of order N, the number of counterparts in $\mathcal{A}$). For the complete enumeration the factor is of order $\sum_{i=1}^{N}\binom{N}{i}$, e.g. 10^{30} for $N = 100$. Second, expected short-fall contributions are ruled out because of their extreme sensitivity to the exposure size (at least for relevant high thresholds).

to the expected individual loss. Finally, the expected individual loss is subtracted

$$ec_A := \frac{e_A l_A}{\sum_{B \in \mathcal{E}} e_B l_B} \left(EC_\gamma - EC_\gamma^{(1)}\right) \qquad (15)$$

to obtain again the necessary requirement $\sum_A ec_A = EC_\gamma$.

In the next section we will focus on the calibration of the models derives in the current and the preceding section. We will restrict the presentation for the "integrated model" to the risk capital (13).

4 Calibration of the models

The capital requirements of both economic models are influenced by parameters that depend on the actual portfolio and those that are portfolio-independent. The calibration estimates the independent parameters first for both models and discusses the portfolio-dependent ingredients afterwards.

We start with the calibration of the integrated model of the previous Section 3. The only obvious portfolio-independent parameter of the variance contribution for the non-performing exposure given in formula (12) is the variance of the "portfolio LGD" σ_Λ^2. We observe the ratio g between the (final) LGD λ_A and the (ex ante) expected LGD l_A. For these ratios we postulate there historical observation for counterpart A in year t to be partly due to the expression of the annual effect Λ_t (we are interested in) and partly due to an ideosyncratic effect ϵ_{At} (we are not interested in). The resulting sparse model is $g_{ti} = \Lambda_t + \epsilon_{ti}$ where we assume the Λ_t's to be independently distributed with expectation 1 and the ϵ_{ti}'s to be independently distributed with expectation 0 for $t = 1, \ldots, T$ and $A \in \mathcal{E}_t$. We define pseudo-observations for Λ_t of $g_{t\cdot} = 1/e_t \sum_{A \in \mathcal{E}_t} e_{At} g_{At}$, weighted by their EAD's e_{At}, where $e_t := \sum_{A \in \mathcal{E}_t} e_{At}$ denotes the portfolio exposure in year t. Assuming sufficiently large samples $\mathcal{E}_t$, the estimate of the σ_Λ^2 is $1/e \sum_{t=1,\ldots,T} e_t (g_{t\cdot} - 1)^2$ (with normalizing $e := \sum_{A \in \mathcal{E}_t} e_t$). Pilot estimates based on 120 losses observed over the seven years 1998-2004 indicate a magnitude of $\hat{\sigma}_\Lambda^2 = 10\%$

There is another hidden portfolio-independent "parameter" for the variance contribution (12) of a non-performing counterpart. The risk contribution to the EC for non-performing exposure depends also on the loss variance $Var(\tilde{L})$ and the EC (see formula (13)). In order to calculate those, we need a decision for the LGD distribution. There is not yet a standard, for the factor Λ as defined in (5) a log-normal distribution (Bürgisser et al. (2001)) implies the possibility of infinite loss rates for a given EAD. And even bimodal distributions are found for LGD distributions (Schuermann (2005)). A generalization of the uniform distribution is the Beta distribution, used by Tasche (2004) for the LGD and in commercial models for the recovery rate, e.g. in CreditMetrics (Gupton et al. (1997)). Based on data of WestLB we found that the generalized Beta distribution fits the distribution of the relative LGD Λ with some modifications. As distribution for the factor Λ we assume an affine transformation of the Beta distribution, i.e.

$$\Lambda \sim a + (b - a)\ Beta(\alpha, \beta),$$

where $0 \leq a < 1 < b$ and $\alpha, \beta > 0$. $Beta(\alpha, \beta)$ denotes the Beta distribution with parameters α and β and density $x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha, \beta)$ with $B(\alpha, \beta) := \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$ and $\Gamma(\alpha) := \int_0^\infty \exp(-x) x^{\alpha-1} dx$.

The assumption $1 = E(\Lambda) = a + (b-a)\ \alpha/(\alpha+\beta)$ forces $\beta = \alpha\ (b-1)/(1-a)$. The parameter β is fixed given a, b and α. For the variance, $\sigma_\Lambda^2 = (b-a)^2 \alpha\beta/((\alpha+\beta)^2(\alpha+\beta+1))$ holds. As mentioned above, the relative LGD Λ has empirically a variance σ_Λ^2 of 0.1, the support of the distribution is $[a = 0.05, b = 2.4]$. The fitted generalized Beta-distribution has parameters $\alpha = 5.3$ and $\beta = 7.8$.

We could calibrate the stand-alone model in Section 2 and especially the provision variance σ_δ^2 independently on the integrated model. However, in order to compare the two models we like to establish a link between σ_Λ^2 and σ_δ^2. Consider the following simplified situation where the models can be compared. Assume that all defaults are settled within the risk horizon of one year (a value close to the average of one-and-a-half years reported earlier in the text for large bank loans). To be precise, consider the case where all defaults occur at the beginning of the year and are settled until the end of the year. Formulated in parameters of the integrated model, the stand-alone model measures the change in provision, i.e. the difference Δ_A between the EAD reduced by the initial provision, $e_A(1 - l_A)$, and the residual exposure after the year, $e_A(1-\lambda_A)$. The final variable is the change divided by the EAD, $\delta_A = \Delta_A/e_A = l_A - \lambda_A = l_A(1-\Lambda)$. The key parameter is the variance $\sigma_\delta^2 = l_A^2 \sigma_\Lambda^2$. Here we see that the economic risk requirements (3) and (13) are only directly comparable for an expected LGD of 100%. However, we can crudely compare capital requirements in the integrated model to those of the stand-alone model with volatility $\sigma_\delta = l_A \sigma_\Lambda$. In order to account for the assumption of one year settlement compared to the more realistic one-and-a-half years, by assuming the LGD to behave like a brownian motion (or at least to have independent increments) we scale the volatility to $\sigma_\delta = l_A \sigma_\Lambda / \sqrt{1.5}$.

In order to estimate the second portfolio-independent parameter ρ, we need a slightly different variable to assess, as compared to the integrated model. Banks usually store relative changes in the provision, δ_{At} (as defined in formula (1)), of their non-performing portfolio $\mathcal{E}_t$ over several years $t = 1, \ldots, T$. The exposure-weighted mean within the portfolio is $\delta_{\cdot t} := 1/e_t \sum_{A \in \mathcal{E}_t} e_{At}\delta_{At} = Y_t + 1/e_t \sum_{A \in \mathcal{E}_t} e_{At}\epsilon_{At}$. Under the assumption of an infinitesimal granular portfolio, i.e. $H \approx 0$, holds $1/e_t \sum_{A \in \mathcal{E}_t} e_{At}\epsilon_{At} \approx 0$ and $Y_t := \delta_{\cdot t}$ is a consistent pseudo-observation for the unobservable Y. Its variance can be estimated by $\hat{\sigma}_Y^2 = 1/e \sum_{t=1,\ldots,T} e_t Y_t^2$ where the weights e_t reflect the difference in portfolio volume over the years. The correlation can be now estimated as

$$\hat{\rho} = \frac{\hat{\sigma}_Y^2 e}{\sum_{t=1,\ldots,T} \sum_{A \in \mathcal{E}_t} e_{At}\delta_{At}^2}.$$

Our loss history with seven pseudo observations reveals a magnitude of 15% for ρ.

All other parameters for the risk contributions depend on the specific portfolio. For the risk contribution in the stand-alone model (3) only the Herfindahl-Hirschmann index H depends on the actual non-performing portfolio. Constructing several non-performing portfolios shows that it can be assumed to range between 0.25% and 2.5%. We will see in the impact study of the following section, that indeed the parameter is almost negligible.

For the risk contribution in the integrated model (see (13) and (12)) we need to specify the "portfolio factor" $EC_\gamma(E(L_p) + \eta)/Var(\tilde{L})$. Let us consider first the relation between the expected loss for the performing portfolio ($E(L_p) = E(\tilde{L}_p)$) and the expected loss for the non-performing portfolio (η). The expected loss of the performing portfolio $\mathcal{A}$ is unconditional, $E(\sum_{A \in \mathcal{A}} \lambda_A e_A I_A)$. Whereas the non-performing portfolio $\mathcal{E}$ is the (conditional) portion of a (former) performing portfolio $\mathcal{A}$ of expected magnitude $\sum_{A \in \mathcal{A}} p_A$ in number and $\sum_{A \in \mathcal{A}} \lambda_A e_A p_A$ in loss. The performing portfolio $\mathcal{A}$ stays essentially the same over time, on the one hand because credit events are rare and on the other hand because banks substitute investments that defaulted with similar investments, to maintain the aimed portfolio composition. Hence $\eta = E(\sum_{A\in\mathcal{E}} \lambda_A e_A) \approx E(E(\sum_{A\in\mathcal{A}} \lambda_A e_A I_A \mid \Lambda)) = \sum_{A\in\mathcal{A}} l_A e_A p_A = E(L_p)$. On balance, we are comfortable with the assumption that the responsibility of the entire expected loss is assigned at equal sizes to both parts of the portfolio, in other words, $E(L_p) + \eta = 2E(L_p)$.

Recall now that the portfolio factor belongs to the risk contribution of a non-performing exposure. Why should that depend e.g. on the degree of diversification in the performing portfolio? Additionally, why should the size of the performing portfolio be important? And in fact, if the exposures of the performing portfolios are scaled with a factor (or are simply displayed in another currency or loss unit), the economic capital, the expected loss and the loss deviation should in a reasonable model scale analogously. It is easy to see that indeed, the factor is unchanged.

We analyse two artificial portfolios for the calibration. First, we specify the performing part of our calibration portfolio. The performing portfolio $\mathcal{A}_{typical}$ we study is - to our knowledge - typical for an international bank. It consists of around 5000 exposures. In order to model concentration risk, the portfolio has the four exposure categories of huge (net) exposures (between 200 million and 1 billion currency units (e.g. Euro)), large exposures (between 30 and 60 million currency units), mediocre exposures (0.3 and 30 million currency units) and small exposures (between 100 and 300 thousand currency units). Our experience suggests a partition of the portfolio into 150 huge exposures (3%), 350 large exposures (7%), 4000 mediocre exposures (80%) and 500 small exposures (10%). Typically, large exposure tend to be associated with a small PD and vice versa. For the huge exposure we assume an expected PD (p_A in model (6)) of 0.03%. For the large exposure the expected PD ranges between 0.03% and 0.07%. The mediocre exposures have expected PD's between 0.07% and 2%, whereas the small exposures' PD expectations can be 2% to 7%. The portfolio is randomly chosen with exposure and PD's from a uniform distribution over the described ranges.

For the non-performing part $\mathcal{E}_{typical}$ one needs only to know the expectation η (see formula (10)). The reasoning half page up suggests it to be approximately the expected loss of $\mathcal{A}_{typical}$.

As a second performing portfolio $\mathcal{A}_{diversified}$ we construct a diversified version of the first portfolio. For the 5000 counterparts the exposure is now (uniform-)randomly selected between 1 and 100 million currency units and the expected PD ranges between 0.03% and 7%. Again, the knowledge about the non-performing part reduces to the expected loss of $\mathcal{A}_{diversified}$.

In order to calculate the loss distribution of our calibration portfolios (using (10)) we need the loss distributions of the performing parts. As an example for a (performing) portfolio model, we use the Panjer-recursion of CreditRisk+ (Credit Suisse First Boston (CSFB) (1997)).

Using CreditRisk+ (at level 99.95%) and the portfolio-independent parameters calibrated in the beginning of this section results in a portfolio factor for the typical portfolio of 11.66 and for the diversified of 11.72. As required, the degree of diversification in the performing portfolio does not change the portfolio factor.

There is another simplification that may be useful for the practitioner. The ratio of economic capital and loss variance is essentially the same for the performing portfolio, as for the entire portfolio. We verify the assumption with the two calibration portfolios. For the typical performing portfolio we find that $EC_{0.9995}/Var(\tilde{L}) = 0.04$ and for the entire portfolio $EC_{0.9995}/Var(\tilde{L}) = 0.035$. For the diversified portfolio the ratios are 0.0026 and 0.0022. The numbers support the simplification of the portfolio factor to $2EC_\gamma(L_p)\ E(L_p)/Var(L_p)$. The simplified portfolio factor for the typical portfolio is 13.48 and for the diversified 13.74, subject to a potential reduction of around 15% to further approximate the correct ones.

5 Impact study

The naive approach for the non-performing portfolio is to consider the LGD after a default to be predictable and add those expectations to the loss of the performing portfolio. The loss distribution of the combined portfolio is simply the loss distribution for the performing portfolio shifted to the right by the sum of all current expected losses for defaulted assets. Expected losses do not have to be covered by capital (in the economic as well as the the current regulatory understanding) and hence non-performing exposure needs no risk capital. The shift to the right ignores the variability of the LGD in the non-performing portfolio and hence the credit Value-at-Risk is underestimated. We compare this standard market practice to our economic models of Sections 2 and 3 and the regulatory requirements in the standardized and the internal-ratings based approach (IRB) approach of Basel II.

On the portfolio level, the model in Section 3 allows us to compare the economic capital for the performing portfolio and the entire portfolio. The calibration portfolios in the preceding section suggest an increase in economic capital of approximately 5%. The risk contributions on the level of single defaulted exposures are the aim of the following study.

For the non-performing portfolio stand-alone, the contribution is given in formula (3) and the derivation for the combined portfolio is found in (13). In order to compare these two risk contributions, we must note that the stand-alone contributions are already net of initial provisions. Contributions for the integrated model are not, hence we assume that the initial provisions are always equal to the expected loss for that specific claim (in-line with the regulatory requirement on provisions) and subtract them from the contribution. Both are linear in the EAD and may hence be expressed as percentages of those in Table 1.

The comparison with regulatory capital is clearly interesting. Regulations on "Past due loans" (Basel Commitee on Banking Supervision (2004), paragraph 75) in the standardized approach declare capital charges for defaulted and unsettled claims. The risk weights (net of specific provision and partial write-offs) are in most cases 100%. Eight percent of the risk weighted assets add to the regulatory capital. Similar to our definition of the EAD, the risk weights apply to the "unsecured portion of any loan". The capital requirement of 8% (relative to the EAD) is considered as a reference (see Table 1).

In the IRB approach of Basel II, paragraph 471 advises to estimate the LGD for any exposure "reflect[ing] the possibility that the bank would have to recognize additional, unexpected losses during the recovery period". How this may be done is explained vaguely: "Appropriate estimates of LGD during periods of high credit losses" are proposed but are liable to misuse by an arbitrary definition of "high". The supplement that "Supervisors will continue to monitor and encourage the development of appropriate approaches to this issue" (Basel Commitee on Banking Supervision (2004), paragraph 468) is a supportive argument for the study at hand. We can quantify the impact of acknowledging the stochastic LGD as included in formula (8). For our two test portfolios $\mathcal{A}_{typical}$ and $\mathcal{A}_{diversified}$ of Section 4 we find that the economic capital (at level 99.95%) is 20% times higher than that of the market standard to use expected LGD's as deterministic forecasts. (The standard deviation of the loss is around 10% higher.) That provides a margin over the expected LGD (dependent on the integrated model in Section 3). The finding that recognizing the stochastic behaviour of the LGD results in 20% higher economic capital can be read as follows. If the LGD for any exposure is taken to be 1.20 times the expected LGD and an internal model is used with deterministic net exposure ($l_{A,\gamma}\times$ EAD), the economic capital is correct. Hence, the 20% surcharge is exactly what the regulator looks for (see Basel Commitee on Banking Supervision (2004), paragraph 272). The resulting capital requirement for defaulted exposure is denoted $l_{A,\gamma} - l_A$ (see Table 1).

– insert Table 1 around here —

The key risk factors for the capital requirements are the volatility and the expectation of the LGD. Interestingly, differences of the economic models become evident. Whereas in the stand-alone model the expectation LGD-correlation ρ measures diversification in the non-performing portfolio, in the integrated model diversification across portfolios is measures in the portfolio factor. In Table 2 we present capital requirements for a variety of risk-factor situations and a confidence level of 99.95%. The correlation ρ is chosen to be 15% (see Section 4). As Schuermann (2005) states that "... recoveries are [...] distributed from 30% to 80%." we investigate expected (individual) LGD between 20% and 70%. Our own estimate of 10% for the LGD variance σ_Λ^2 (see again Section 4) is the center of the range 7-13%.

– insert Table 2 around here —

Evidently, the regulatory capital needed to cover the non-performing portfolio is less risk sensitive than our formulae for the economic capital. The standardized approach is the least adaptive. The IRB approach adapts for the expected LGD, in the sense that more expected LGD requires more risk capital. The integrated methods adapts for both expected LGD and LGD volatility. Interestingly, a small LGD volatility results in a negligible capital charges. The stand-alone model adapts only for the LGD volatility explicitly, and for the LGD expectation implicitly via the the calibration of σ_δ^2. Interestingly, the stand-alone model depends on the LGD volatility, whereas the integrated model depends on the variance. The immediate suggestion is that the integrated model results in less sensitivity of the capital requirement with respect to the LGD variability because the quadratic function is contracting for arguments between 0 and 1. However, the examples show the contrary picture, the reason being that the factors invert the relation. The requirements are less spread for the stand-alone model than for the integrated model. On average the level of the requirement of the stand-alone model is higher than that of the integrated model (as well as for the regulatory requirements). We suspect that the lack in diversification potential with the performing-portfolio is the reason leading to overstatement of the capital requirements on the overall portfolio level.

However, some comparisons can still be drawn. One can see that the sensitivity of the stand-alone charge to the LGD volatility is smaller than that of the integrated charge. An additional finding for the stand-alone risk contributions is the very small sensitivity with respect to the portfolio concentration H. This might be a disadvantage for the active portfolio management, as little incentive for diversification results.

6 Conclusion

We propose two methods to calculate economic risk contributions for non-performing exposure in portfolio credit risk and compared them to regulatory capital requirements. The economic view is significantly more risk sensitive with respect to the LGD volatility. However, the two models are quite different. The stand-alone model - with its focus on annual changes - models the internal process of provisioning in-line with the risk horizon. The integrated model keeps the over nominal level at the expense of a time-independent model for the LGD. Both models result in capital charges that depend on the actual portfolio composition. The stand-alone model has a concentration penalty, although being negligible but in extreme portfolios. The integrated model depends on the entire portfolio with the advantage of allowing for diversification. The dependence of the capital charges on the expectation of the LGD is economically not clear, it might be an algebraic artefact. On the other hand, the regulatory IRB approach postulates that dependence as well. Maybe the aim to separate first and second moments at all levels of the loss distribution is too ambitious.

Also important for the model choice is the data requirement for the calibration. Although we believe that portfolio owners have all the information at hand to calibrate both, the lack of some information may lead to a rejection of one or the other.

From a stochastic point of view we see the restriction of modeling with random variables (on $\mathbb{R}$), rather than with stochastic processes, although the stand-alone model is a step in that direction. The order condition that settlement of claim follows a default calls for time as a co-variable. However, stochastic processes appear to be overly technical to accomplish similar brought and practicably feasible results. Some aspects - as the relation between expected losses in performing and non-performing portfolio - should be treated with those models.

At last we like to draw the attention to a particularly interesting point. The randomness of the LGD is motivated mainly by the variability of the collateral's value. In that sense, our study is an attempt to integrate market risk in (portfolio) credit risk quantification. One must consider, whether risk capital is not double-counted in the current banking set-up of separate market and credit risk management.

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Table 1: Capital requirement formulae for economic and regulatory capital.

Model	Capital requirement
Economic risk	
Stand-alone	$u_\gamma (H+\rho)^{1/2}\sigma_\delta$
Integrated model	$\left(\frac{EC_\gamma E(\tilde{L})}{Var(\tilde{L})}\sigma_\Lambda^2 - 1\right) l_A$
Basel II	
Standardized approach	8%
IRB	$l_{A,\gamma} - l_A$

Table 2: Capital requirements relative to the EAD for non-performing exposure dependent on LGD expectation and the variance of the LGD σ_Λ^2: Comparison of stand-alone method (with Herfindahl-Hirschmann index $H = 1\%$ and correlation $\rho = 15\%$), integrated method, IRB approach and the standardized approach of Basel II.

Risk factor		Model			
		Economic capital		Basel II	
E(LGD)	Var(LGD)	Stand-alone$^{H=0.25\%}_{H=2.5\%}$	Integrated	IRB	Standardized
20%	7%	$5.7\%^{-0.1}_{+0.2}$	0%	4%	8%
	10%	$6.8\%^{-0.1}_{+0.3}$	3%	4%	8%
	13%	$7.8\%^{-0.2}_{+0.3}$	10%	4%	8%
45%	7%	$12.8\%^{-0.3}_{+0.6}$	0%	9%	8%
	10%	$15.3\%^{-0.3}_{+0.7}$	7%	9%	8%
	13%	$17.4\%^{-0.4}_{+0.8}$	23%	9%	8%
70%	7%	$19.9\%^{-0.5}_{+0.9}$	0%	14%	8%
	10%	$23.8\%^{-0.6}_{+1.1}$	11%	14%	8%
	13%	$27.1\%^{-0.6}_{+1.3}$	35%	14%	8%