Simulation and option pricing in Lévy copula models

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Abstract

Lévy copulas, introduced in [8], are functions that completely characterize the law of a multidimensional Lévy process given the laws of its components. In this paper, after recalling the basic properties of Lévy copulas, we discuss the simulation of multidimensional Lévy processes with dependence structure given by a Lévy copula. Being able to describe the dependence structure of a Lévy process in terms of its Lévy copula allows us to quantify the effect of dependence on the prices of basket options in a multidimensional exponential Lévy model. We conclude that these prices are highly sensitive not only to the linear correlation between assets but also to the exact type of dependence beyond linear correlation.

Key words: correlation, dependence, Lévy copulas, multi-asset options, simulation

1 Introduction

Many financial applications require a multidimensional model with jumps, taking into account the dependence between components. While Lévy processes have been successfully applied by many authors to construct one-dimensional jump models (cf. e.g. [2, 5, 9, 10, 15]), multivariate applications continue to be dominated by Brownian motion (but cf. [12] in this respect). To fill this gap, the notion of Lévy copula was introduced in [8] (see also Chapter 5 in [4]).

A Lévy copula allows to describe in a time-dependent fashion the dependence structure of a Lévy process without Gaussian component. On the

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other hand, given n one-dimensional Lévy processes $X^1, \ldots, X^n$, Lévy copulas allow to characterize all n-dimensional Lévy processes whose components have the same laws as $X^1, \ldots, X^n$. In Section 3 we recall the definition of Lévy copula and the main theorem, explaining the relation between Lévy copulas and Lévy processes.

Lévy copulas turn out to be a convenient tool for simulating multidimensional Lévy processes with specified dependence. In Section 4 we prove two theorems which show how multidimensional Lévy processes with dependence structures given by Lévy copulas can be simulated in the finite variation case (Theorem 4.3) and in the infinite variation case (Theorem 4.4).

Section 5 discusses the applications of Lévy copulas to multi-asset option pricing. We construct a two-dimensional exponential Lévy model with variance gamma margins and compute the prices of two types of multi-asset options using the Monte Carlo method. Choosing different sets of dependence parameters corresponding to the same correlation level enables us to quantify the sensitivity of prices to the exact type of dependence beyond linear correlation.

2   Lévy processes

In this section we recall the essential properties of Lévy processes. The reader can consult [16] or [4] for details.

A Lévy process $(X_t)_{t \geq 0}$ is a càdlàg stochastic process with stationary independent increments, satisfying $X_0 = 0$. The characteristic function of an $\mathbb{R}^d$-valued Lévy process has the following form, called the Lévy-Khintchine representation [16]:

$$E[e^{i\langle z, X_t \rangle}] = e^{t\psi(z)}, \quad \text{with}$$

$$\psi(z) = -\frac{1}{2}\langle z, Az \rangle + i\langle \gamma, z \rangle + \int_{\mathbb{R}^d}(e^{i\langle z, x \rangle} - 1 - i\langle z, x \rangle 1_{|x| \leq 1})\nu(dx), \quad (2.1)$$

where A is a symmetric nonnegative-definite $d \times d$ matrix (the unit covariance matrix of the Brownian motion part of the Lévy process), $\gamma \in \mathbb{R}^d$ and ν is a positive measure on $\mathbb{R}^d$ verifying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\nu(dx) < \infty.$$

The triplet (A, ν, γ) is called the characteristic triplet of X.

If the Lévy measure satisfies $\int_{\mathbb{R}^d}(|x| \wedge 1)\nu(dx) < \infty$, (this means that the jump part of the Lévy process is of finite variation) one does not need to truncate small jumps in (2.1) and the Lévy-Khintchine representation can be rewritten as

$$\psi(z) = -\frac{1}{2}\langle z, Az \rangle + i\langle b, z \rangle + \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1)\nu(dx). \qquad (2.2)$$

The vector b is in this case called *drift* of the process X.

Example 2.1. The variance gamma process [3, 10] is a one-dimensional Lévy process without Gaussian component ($A = 0$). It is obtained by time-changing a Brownian motion with drift with a gamma process and has the characteristic exponent of the form:

$$\psi(u) = ibu - \frac{1}{\kappa}\log(1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u). \qquad (2.3)$$

The Lévy measure of the variance gamma process has a density given by

$$\nu(x) = \frac{c}{|x|}e^{-\lambda_-|x|}1_{x<0} + \frac{c}{x}e^{-\lambda_+x}1_{x>0}, \qquad (2.4)$$

where $c = 1/\kappa$, $\lambda_+ = \frac{\sqrt{\theta^2+2\sigma^2/\kappa}}{\sigma^2} - \frac{\theta}{\sigma^2}$ and $\lambda_- = \frac{\sqrt{\theta^2+2\sigma^2/\kappa}}{\sigma^2} + \frac{\theta}{\sigma^2}$.

In the same way as the law of a random vector can be represented by its distribution function, the Lévy measure of a Lévy process can be represented by its tail integral.

Definition 2.1. Let X be a $\mathbb{R}^d$-valued Lévy process with Lévy measure ν. The *tail integral* of X is the function $U : (\mathbb{R} \setminus \{0\})^d \to \mathbb{R}$ defined by

$$U(x_1, \ldots, x_d) := \prod_{i=1}^d \operatorname{sgn}(x_i)\nu\left(\prod_{j=1}^d \mathcal{I}(x_j)\right),$$

where for every $x \in \mathbb{R}$,

$$\mathcal{I}(x) := \begin{cases} [x, \infty), & x \geq 0, \\ (-\infty, x), & x < 0. \end{cases} \qquad (2.5)$$

For a nonempty set $I \subset \{1, \ldots, d\}$, the *I-marginal tail integral* U^I of X is the tail integral of the process $X^I := (X^i)_{i \in I}$. To simplify notation, we denote one-dimensional margins by $U_i := U^{\{i\}}$. The Lévy measure of a Lévy

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process X is completely determined by its tail integral and all its marginal tail integrals (cf. Lemma 3.5 in [8]).

We now briefly recall the definition of a Poisson random measure and the Lévy-Itô decomposition of the sample paths of Lévy processes, which are essential for Section 4 dealing with the simulation of Lévy processes.

Let μ be a σ-finite positive measure on $\mathbb{R}^d$ endowed with its Borel σ-field $\mathcal{B}(\mathbb{R}^d)$. A Poisson random measure with intensity measure μ is an integer-valued random measure M such that

1. For every random element ω, $M(\cdot, \omega)$ is a measure on $\mathbb{R}^d$.

2. For every $A \in \mathcal{B}(\mathbb{R}^d)$, $M(A)$ is a Poisson random variable with mean $\mu(A)$.

3. If $A_1, \ldots, A_n$ are disjoint then $M(A_1), \ldots, M(A_n)$ are independent.

Let X be an $\mathbb{R}^d$-valued Lévy process with characteristic triplet (A, ν, γ). The Lévy-Itô decomposition theorem [16] states that there exist a Brownian motion $(B_t)_{t \geq 0}$ with covariance matrix A and a Poisson random measure J_X on $[0, \infty) \times \mathbb{R}^d$ with intensity measure $dt \times \nu$, such that the sample paths of X can be represented as follows:

$$X_t = \gamma t + B_t + X_t^l + \lim_{\varepsilon \downarrow 0} \tilde{X}_t^\varepsilon, \quad \text{where} \qquad (2.6)$$

$$X_t^l = \int_{|x| \geq 1, s \in [0,t]} x J_X(ds \times dx) \quad \text{and}$$

$$\tilde{X}_t^\varepsilon = \int_{\varepsilon \leq |x| < 1, s \in [0,t]} x\{J_X(ds \times dx) - ds \times \nu(dx)\}$$

The terms in (2.6) are independent and the convergence in the last term is almost sure and uniform in t on $[0, T]$.

If the Lévy measure satisfies $\int_{\mathbb{R}^d} (|x| \wedge 1)\nu(dx) < \infty$, truncation of small jumps is not needed and Equation (2.6) simplifies to

$$X_t = bt + B_t + \int_{[0,t] \times \mathbb{R}^d} x J_X(ds \times dx) \qquad (2.7)$$

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3 Lévy copulas

We start by recalling a few facts on increasing functions. We set $\overline{\mathbb{R}} := (-\infty, \infty]$ in this paper and

$$\operatorname{sgn} x := \begin{cases} 1 & \text{for } x \geq 0 \\ -1 & \text{for } x < 0. \end{cases}$$

For $a, b \in \overline{\mathbb{R}}^d$ we write $a \leq b$ if $a_k \leq b_k$, $k = 1, \ldots, d$. In this case, let $(a, b]$ denote a right-closed left-open interval of $\overline{\mathbb{R}}^d$:

$$(a, b] := (a_1, b_1] \times \cdots \times (a_d, b_d].$$

Definition 3.1. Let $F : S \to \overline{\mathbb{R}}$ for some subset $S \subset \overline{\mathbb{R}}^d$. For $a, b \in S$ with $a \leq b$ and $\overline{(a, b]} \subset S$, the F-*volume* of $(a, b]$ is defined by

$$V_F((a, b]) := \sum_{u \in \{a_1, b_1\} \times \cdots \times \{a_d, b_d\}} (-1)^{N(u)} F(u),$$

where $N(u) := \#\{k : u_k = a_k\}$.

F is called *d-increasing* if $V_F((a, b]) \geq 0$ for all such $a, b \in S$.

For example, for $d = 2$ we obtain

$$V_F((a_1, b_1] \times (a_2, b_2]) = F(a_1, b_1) - F(a_1, b_2) - F(a_2, b_1) + F(a_2, b_2).$$

The distribution function F of a random variable provides an example of an increasing function (the F-volume of a rectangle is in this case equal to the probability that the random variable belongs to this rectangle).

Definition 3.2. Let $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ be a *d-increasing* function such that $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$. For any non-empty index set $I \subset \{1, \ldots, d\}$, the *I-margin* of F is the function $F^I : \overline{\mathbb{R}}^I \to \overline{\mathbb{R}}$, defined by

$$F^I((u_i)_{i \in I}) := \lim_{c \to \infty} \sum_{(u_j)_{j \in I^c} \in \{-c, \infty\}^{I^c}} F(u_1, \ldots, u_d) \prod_{j \in I^c} \operatorname{sgn} u_j,$$

where $I^c := \{1, \ldots, d\} \setminus I$.

The stage is now set to give the definition of a Lévy copula. The properties of a Lévy copula are similar to those of an ordinary copula (see [11] for an introduction to copulas) but the domain of definition is completely different; this is due to the fact that Lévy measures are not necessarily finite measures.

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Definition 3.3. A function $F : \overline{\mathbb{R}}^d \to \overline{\mathbb{R}}$ is called *Lévy copula* if

1. $F(u_1, \ldots, u_d) \neq \infty$ for $(u_1, \ldots, u_d) \neq (\infty, \ldots, \infty)$,

2. $F(u_1, \ldots, u_d) = 0$ if $u_i = 0$ for at least one $i \in \{1, \ldots, d\}$,

3. F is d-increasing,

4. $F^{\{i\}}(u) = u$ for any $i \in \{1, \ldots, d\}$, $u \in \mathbb{R}$.

Example 3.1 *(Clayton Lévy copulas).* Let $d = 2$ and define

$$F(u, v) = (|u|^{-\theta} + |v|^{-\theta})^{-1/\theta} \left(\eta 1_{\{uv \geq 0\}} - (1 - \eta) 1_{\{uv < 0\}} \right). \tag{3.1}$$

In [8] it is shown that F is a Lévy copula for any $\theta > 0$ and $\eta \in [0, 1]$. In this family of Lévy copulas, the parameter η determines the dependence of the sign of jumps: when $\eta = 1$, the two components always jump in the same direction, and when $\eta = 0$, positive jumps in one component are accompanied by negative jumps in the other and vice versa. The parameter θ is responsible for the dependence of absolute values of jumps in different components. In particular, if $\eta = 1$ and $\theta \to 0$, the two components become independent and the case $\eta = 1$ and $\theta \to \infty$ corresponds to complete dependence.

The following result, established in [8], clarifies the relation between Lévy copulas and Lévy processes.

Theorem 3.1. *Let $X = (X^1, \ldots, X^d)$ be a $\mathbb{R}^d$-valued Lévy process. Then there exists a Lévy copula F such that the tail integrals of X satisfy:*

$$U^I((x_i)_{i \in I}) = F^I((U_i(x_i))_{i \in I}) \tag{3.2}$$

for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. The Lévy copula F is unique on $\prod_{i=1}^{d} \overline{\operatorname{Ran} U_i}$.

Conversely, let F be a d-dimensional Lévy copula and U_i, $i = 1, \ldots, d$ tail integrals of real-valued Lévy processes. Then there exists an $\mathbb{R}^d$-valued Lévy process X whose components have tail integrals $U_1, \ldots, U_d$ and whose marginal tail integrals satisfy Equation (3.2) for any non-empty $I \subset \{1, \ldots, d\}$ and any $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^I$. The Lévy measure ν of X is uniquely determined by F and U_i, $i = 1, \ldots, d$.

To construct an n-dimensional Lévy model, one can thus take n one-dimensional Lévy processes (e.g. variance gamma, but different components can also be of completely different nature, say, a compound Poisson component and an infinite intensity one) and one Lévy copula, possibly from a parametric family. This is the approach taken in Section 5 to construct a two-dimensional exponential Lévy model with variance gamma margins.

4 Simulation of multidimensional dependent Lévy processes

To simulate a Lévy process $(X_t)_{0 \le t \le 1}$ on $\mathbb{R}^d$ with Lévy measure ν, our strategy is first to simulate a Poisson random measure on $[0, 1] \times \mathbb{R}^d$ with intensity measure $dt \times \nu$. The Lévy process can then be constructed via the Lévy-Itô decomposition (2.6).

Let F be a Lévy copula such that for every $I \in \{1, \dots, d\}$ nonempty,

$$\lim_{(x_i)_{i \in I} \to \infty} F(x_1, \dots, x_d) = F(x_1, \dots, x_d)|_{(x_i)_{i \in I} = \infty}. \tag{4.1}$$

This Lévy copula defines a positive measure μ on $\mathbb{R}^d$ with Lebesgue margins such that for each $a, b \in \mathbb{R}^d$ with $a \le b$,

$$V_F((a, b]) = \mu((a, b]). \tag{4.2}$$

In the following technical lemma, needed in the sequel, we establish the relation between μ and the Lévy measures of processes having F as their Lévy copula. For a one-dimensional tail integral U, the (generalized) inverse tail integral $U^{(-1)}$ is defined by

$$U^{(-1)}(u) := \begin{cases} \sup\{x > 0 : U(x) \ge u\} \vee 0, & u \ge 0 \\ \sup\{x < 0 : U(x) \ge u\}, & u < 0. \end{cases} \tag{4.3}$$

Lemma 4.1. *Let ν be a Lévy measure on $\mathbb{R}^d$ with marginal tail integrals U_i, $i = 1, \dots, d$, and Lévy copula F satisfying (4.1), let μ be defined by (4.2) and let*

$$f : (u_1, \dots, u_d) \mapsto (U_1^{(-1)}(u_1), \dots, U_d^{(-1)}(u_d)).$$

Then ν is the image measure of μ by f.

Proof. We must prove that for each $A \in \mathcal{B}(\mathbb{R}^d)$,

$$\nu(A) = \mu(\{u \in \mathbb{R}^d : f(u) \in A\}),$$

but because ν is completely determined by the set of all its marginal tail integrals (Lemma 3.5 in [8]), it is sufficient to show that for each $I \subset \{1, \dots, d\}$ nonempty and for all $(x_i)_{i \in I} \in (\mathbb{R} \setminus \{0\})^{|I|}$,

$$U^I((x_i)_{i \in I}) = \mu(\{u \in \mathbb{R}^d : U_i^{(-1)}(u_i) \in \mathcal{I}(x_i), i \in I\}),$$

where $\mathcal{I}(x)$ was defined in (2.5). However, since U_i is left-continuous, for every i, $U_i^{(-1)}(u) \in \mathcal{I}(x)$ if and only if $u \in (U_i(x) \wedge 0, U_i(x) \vee 0]$. Therefore,

$$\mu(\{u \in \mathbb{R}^d : U_i^{(-1)}(u_i) \in \mathcal{I}(x_i), i \in I\})$$
$$= \mu(\{u \in \mathbb{R}^d : u_i \in (U_i(x_i) \wedge 0, U_i(x_i) \vee 0], i \in I\}) = F^I((U_i(x_i))_{i \in I}),$$

and an application of Theorem 3.1 completes the proof. $\qquad\square$

In Theorems 4.3 and 4.4 below, to simulate the jumps of a multidimensional Lévy process (more precisely, of the corresponding Poisson random measure), we will first simulate the jumps in the first component, and then the jumps in the other components conditionally on the jumps in the first one. We therefore proceed by analyzing the conditional distributions of μ. By Theorem 2.28 in [1], there exists a family, indexed by $\xi \in \mathbb{R}$, of positive Radon measures $K(\xi, dx_2 \cdots dx_d)$ on $\mathbb{R}^{d-1}$, such that

$$\xi \mapsto K(\xi, dx_2 \cdots dx_d)$$

is Borel measurable and

$$\mu(dx_1 \ldots dx_d) = dx_1 \times K(x_1, dx_2 \cdots dx_d). \tag{4.4}$$

In addition, $K(\xi, \mathbb{R}^{d-1}) = 1$ almost everywhere, that is, $K(\xi, \cdot)$ is, almost everywhere, a probability distribution. In the sequel we will call $\{K(\xi, \cdot)\}_{\xi \in \mathbb{R}}$ the family of conditional probability distributions associated with Lévy copula F.

Let F_ξ be the distribution function of the measure $K(\xi, \cdot)$:

$$F_\xi(x_2, \ldots, x_d) := K(\xi, (-\infty, x_2] \times \cdots \times (-\infty, x_d]). \tag{4.5}$$

The following lemma shows that it can be computed in a simple manner from the Lévy copula F.

Lemma 4.2. *Let F be a Lévy copula satisfying (4.1), and F_ξ be the corresponding conditional distribution function, defined by (4.5). Then, there exists a set $N \subset \mathbb{R}$ of zero Lebesgue measure such that for every fixed $\xi \in \mathbb{R} \backslash N$, $F_\xi(\cdot)$ is a probability distribution function, satisfying*

$$F_\xi(x_2, \ldots, x_d)$$
$$= \operatorname{sgn}(\xi) \frac{\partial}{\partial \xi} V_F((\xi \wedge 0, \xi \vee 0] \times (-\infty, x_2] \times \cdots \times (-\infty, x_d]) \tag{4.6}$$

in every point $(x_2, \ldots, x_d)$, where F_ξ is continuous.

Remark 4.1. Since the law of a random variable is completely determined by the values of its distribution function at the continuity points of the latter, being able to compute F_ξ at all points where it is continuous is sufficient for all practical purposes.

Proof. Since it has already been observed that $K(\xi, \mathbb{R}^{d-1}) = 1$ almost everywhere, we only need to prove the second part of the lemma. Let

$$G(x_1, \ldots, x_d) := \operatorname{sgn} x_1 V_F((x_1 \wedge 0, x_1 \vee 0] \times (-\infty, x_2] \times \cdots \times (-\infty, x_d])$$

By Theorem 2.28 in [1], for each $f \in L^1(\mathbb{R}^d, \mu)$,

$$\int_{\mathbb{R}^d} f(x_1, \ldots, x_d) \mu(dx_1 \cdots dx_d)$$
$$= \int_{-\infty}^{\infty} dx_1 \int_{\mathbb{R}^d} f(x_1, \ldots, x_d) K(x_1, dx_2 \cdots dx_d), \quad (4.7)$$

which implies that

$$G(x_1, \ldots, x_d) = \operatorname{sgn} x_1 \int_{(x_1 \wedge 0, x_1 \vee 0]} d\xi F_\xi(x_2, \ldots, x_d),$$

Therefore, for fixed $(x_2, \ldots, x_d)$, (4.6) holds ξ-almost everywhere. Since a union of countably many sets of zero measure is again a set of zero measure, there exists a set $N \subset \mathbb{R}$ of zero Lebesgue measure such that for every $\xi \in \mathbb{R} \setminus N$, (4.6) holds for all $(x_2, \ldots, x_d) \in \mathbb{Q}^d$, where $\mathbb{Q}$ denotes the set of rational numbers.

Fix $\xi \in \mathbb{R} \setminus N$ and let $x \in \mathbb{R}^{d-1}$ and $\{x_n^+\}$ and $\{x_n^-\}$ be two sequences of $d-1$-dimensional vectors with coordinates in $\mathbb{Q}$, converging to x from above and from below (componentwise). Since F_ξ is increasing in each coordinate (as a probability distribution function), the limits $\lim_n F_\xi(x_n^+)$ and $\lim_n F_\xi(x_n^-)$ exist. Suppose that

$$\lim_n F_\xi(x_n^+) = \lim_n F_\xi(x_n^-) = F^* \qquad (4.8)$$

and observe that for every $\delta \neq 0$,

$$\frac{G(\xi + \delta, x_n^-) - G(\xi, x_n^-)}{\delta} \leq \frac{G(\xi + \delta, x) - G(\xi, x)}{\delta} \leq \frac{G(\xi + \delta, x_n^+) - G(\xi, x_n^+)}{\delta}.$$

For every $\varepsilon > 0$, in view of (4.8), there exists N_0 such that for every $n \geq N_0$, $F_\xi(x_n^+) - F^* \leq \varepsilon/2$ and $F^* - F_\xi(x_n^-) \leq \varepsilon/2$. Since G is differentiable with

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respect to the first variable at points (ξ, x_n^+) and (ξ, x_n^-), we can choose δ small enough so that

$$\left| \frac{G(\xi + \delta, x_n^-) - G(\xi, x_n^-)}{\delta} - F_\xi(x_n^-) \right| \leq \varepsilon/2$$

and

$$\left| \frac{G(\xi + \delta, x_n^+) - G(\xi, x_n^+)}{\delta} - F_\xi(x_n^+) \right| \leq \varepsilon/2$$

This proves that

$$\lim_{\delta \to 0} \frac{G(\xi + \delta, x) - G(\xi, x)}{\delta} = F^*.$$

We have thus shown that F_ξ satisfies Equation (4.6) in all points where (4.8) holds, that is, where F_ξ is continuous. $\qquad\square$

In the following two theorems we show how Lévy copulas may be used to simulate multidimensional Lévy processes with specified dependence. Our results can be seen as an extension to Lévy processes, represented by Lévy copulas, of the series representation results, developed by Rosinski and others (see [14] and references therein). The first result concerns the simpler case when the Lévy process has finite variation on compacts.

Theorem 4.3. (Simulation of multidimensional Lévy processes, finite variation case)
Let ν be a Lévy measure on $\mathbb{R}^d$, satisfying $\int (|x| \wedge 1)\nu(dx) < \infty$, with marginal tail integrals U_i, $i = 1, \ldots, d$ and Lévy copula $F(x_1, \ldots, x_d)$, such that the condition (4.1) is satisfied, and let $K(x_1, dx_2 \cdots dx_d)$ be the corresponding conditional probability distributions, defined by (4.5). Let $\{V_i\}$ be a sequence of independent random variables, uniformly distributed on $[0, 1]$. Introduce d random sequences $\{\Gamma_i^1\}, \ldots, \{\Gamma_i^d\}$, independent from $\{V_i\}$ such that

- *$N = \sum_{i=1}^{\infty} \delta_{\{\Gamma_i^1\}}$ is a Poisson random measure on $\mathbb{R}$ with Lebesgue intensity measure.*

- *Conditionally on Γ_i^1, the random vector $(\Gamma_i^2, \ldots, \Gamma_i^d)$ is independent from Γ_j^k with $j \neq i$ and all k and is distributed on $\mathbb{R}^{d-1}$ with law $K(\Gamma_i^1, dx_2 \cdots dx_d)$.*

Then

$$(Z_t)_{0 \leq t \leq 1} \quad \text{where} \quad Z_t^k = \sum_{i=1}^{\infty} U_i^{(-1)}(\Gamma_i^k) 1_{[0,t]}(V_i), \quad k = 1, \ldots, d, \quad (4.9)$$

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is a Lévy process on the time interval $[0, 1]$ *with characteristic function*

$$E\left[e^{i\langle u, Z_t\rangle}\right] = \exp\left(t\int_{\mathbb{R}^d}(e^{i\langle u,z\rangle} - 1)\nu(dz)\right). \qquad (4.10)$$

Remark 4.2. The probability distribution function of $(\Gamma_i^2, \ldots, \Gamma_i^d)$ conditionally on Γ_i^1 is known from Lemma 4.6.

Remark 4.3. The sequence $\{\Gamma_i^1\}_{i \geq 1}$ can be constructed, for example, as follows. Let $\{X_i\}_{i \geq 1}$ be a sequence of jump times of a Poisson process with jump intensity equal to 2. Then it is easy to check that one can define Γ_i^1 by $\Gamma_i^1 = X_i(-1)^i$.

Proof. First note that $\{\Gamma_i^k\}$ are well defined since by Lemma 4.2, $K(x_1, \cdot)$ is a probability distribution for almost all x_1. Let

$$Z_{\tau,t}^k = \sum_{-\tau \leq \Gamma_i^1 \leq \tau} U_k^{(-1)}(\Gamma_i^k)1_{V_i \leq t}, \quad k = 1, \ldots, d.$$

By Proposition 3.8 in [13],

$$Z_{\tau,t}^k = \int_{[0,t]\times[-\tau,\tau]\times\mathbb{R}^{d-1}} U_k^{(-1)}(x_k)M(ds \times dx_1 \cdots dx_d),$$

where M is a Poisson random measure on $[0, 1] \times \mathbb{R}^d$ with intensity measure $dt \times \mu(dx_1 \cdots dx_d)$, and the measure μ was defined in Equation (4.2).

By Lemma 4.1 and Proposition 3.7 in [13],

$$Z_{\tau,t}^k = \int_{[0,t]\times\mathbb{R}^d} x_k N_\tau(ds \times dx_1 \cdots dx_d), \qquad (4.11)$$

for some Poisson random measure N_τ on $[0, 1] \times \mathbb{R}^d$ with intensity measure $ds \times \nu_\tau(dx_1 \cdots dx_d)$, where

$$\nu_\tau := 1_{(-\infty, U_1^{(-1)}(-\tau)]\cup[U_1^{(-1)}(\tau),\infty)}(x_1)\nu(dx_1 \cdots dx_d) \qquad (4.12)$$

The Lévy-Itô decomposition (2.7) implies that $Z_{\tau,t}$ is a Lévy process on the time interval $[0, 1]$ with characteristic function

$$E\left[e^{i\langle u, Z_{\tau,t}\rangle}\right] = \exp\left(t\int_{\mathbb{R}^d}(e^{i\langle u,z\rangle} - 1)\nu_\tau(dz)\right).$$

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Let h be a bounded continuous function such that $h(x) \equiv x$ on a neighborhood of 0. Since $\lim_{\tau \to \infty} U_1^{(-1)}(\tau) = 0$ and $\lim_{\tau \to \infty} U_1^{(-1)}(-\tau) = 0$, by dominated convergence,

$$\int_{\mathbb{R}^d} h^2(x) \nu_\tau(dx) \xrightarrow[\tau \to \infty]{} \int_{\mathbb{R}^d} h^2(x) \nu(dx)$$

$$\text{and} \qquad \int_{\mathbb{R}^d} h(x) \nu_\tau(dx) \xrightarrow[\tau \to \infty]{} \int_{\mathbb{R}^d} h(x) \nu(dx).$$

Moreover, for every $f \in C_b(\mathbb{R}^d)$ such that $f(x) \equiv 0$ on a neighborhood of 0,

$$\int_{\mathbb{R}^d} f(x) \nu_\tau(dx) = \int_{\mathbb{R}^d} f(x) \nu(dx)$$

starting from sufficiently large τ. Therefore, Corollary VII.3.6 in [7] allows to conclude that $(Z_{\tau,t})_{0 \leq t \leq 1}$ converges in law to a Lévy process with characteristic function given by (4.10). $\qquad \square$

If the Lévy process has paths of infinite variation on compacts, it can no longer be represented as the sum of its jumps and we have to introduce a centering term into the series (4.9).

Theorem 4.4. (Simulation of multidimensional Lévy processes, infinite variation case)
Let ν be a Lévy measure on $\mathbb{R}^d$ with marginal tail integrals U_i, $i = 1, \ldots, d$ and Lévy copula $F(x_1, \ldots, x_d)$, such that the condition (4.1) is satisfied. Let $\{V_i\}$ and $\{\Gamma_i^1\}, \ldots, \{\Gamma_i^d\}$ be as in Theorem 4.3. Let

$$A_k(\tau) = \int_{|x| \leq 1} x_k \nu_\tau(dx_1 \cdots dx_d), \quad k = 1 \ldots d,$$

where ν_τ is given by (4.12). Then the process

$$(Z_{\tau,t})_{0 \leq t \leq 1}, \quad \text{where} \quad Z_{\tau,t}^k = \sum_{-\tau \leq \Gamma_i^1 \leq \tau} U_k^{(-1)}(\Gamma_i^k) 1_{V_i \leq t} - t A_k(\tau),$$

converges in law as $\tau \to \infty$ to a Lévy process $(Z_t)_{0 \leq t \leq 1}$ on the time interval $[0, 1]$ with characteristic function

$$E\left[e^{i\langle u, Z_t \rangle}\right] = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle u, z \rangle} - 1 - i\langle u, z \rangle) 1_{|z| \leq 1} \nu(dz) \right). \qquad (4.13)$$

Proof. The proof is essentially the same as in Theorem 4.3. Similarly to Equation (4.11), $Z_{\tau,t}^k$ can now be represented as

$$Z_{\tau,t}^k = \int\limits_{[0,t] \times \{x \in \mathbb{R}^d : |x| \leq 1\}} x_k \{N_\tau(ds \times dx_1 \cdots dx_d) - ds\nu_\tau(dx_1 \cdots dx_d)\}$$

$$+ \int_{[0,t] \times \{x \in \mathbb{R}^d : |x| > 1\}} x_k N_\tau(ds \times dx_1 \cdots dx_d),$$

where N_τ is a Poisson random measure on $[0,1] \times \mathbb{R}^d$ with intensity measure $ds \times \nu_\tau$, and ν_τ is defined by (4.12). This entails that $(Z_{\tau,t})$ is a Lévy process (compound Poisson) with characteristic function

$$E\left[e^{i\langle u, Z_{\tau,t}\rangle}\right] = \exp\left(t \int_{\mathbb{R}^d} (e^{i\langle u,z\rangle} - 1 - i\langle u,z\rangle 1_{|z|\leq 1})\nu_\tau(dz)\right).$$

Corollary VII.3.6 in [7] once again allows to conclude that $(Z_{\tau,s})_{0\leq s\leq 1}$ converges in law to a Lévy process with characteristic function (4.13). $\qquad\square$

Example 4.1. Let $d = 2$ and F be the Lévy copula of Example 3.1. A straightforward computation yields:

$$F_\xi(x_2) = \left\{(1-\eta) + \left(1 + \left|\frac{\xi}{x_2}\right|^\theta\right)^{-1-1/\theta}(\eta - 1_{x_2<0})\right\} 1_{\xi\geq 0}$$

$$+ \left\{\eta + \left(1 + \left|\frac{\xi}{x_2}\right|^\theta\right)^{-1-1/\theta}(1_{x_2\geq 0} - \eta)\right\} 1_{\xi<0}. \quad (4.14)$$

This conditional distribution function can be inverted analytically:

$$F_\xi^{-1}(u) = B(\xi,u)|\xi|\left\{C(\xi,u)^{-\frac{\theta}{\theta+1}} - 1\right\}^{-1/\theta}$$

$$\text{with} \quad B(\xi,u) = \text{sgn}(u - 1 + \eta)1_{\xi\geq 0} + \text{sgn}(u - \eta)1_{\xi<0}$$

$$\text{and} \quad C(\xi,u) = \left\{\frac{u - 1 + \eta}{\eta}1_{u\geq 1-\eta} + \frac{1-\eta-u}{1-\eta}1_{u<1-\eta}\right\} 1_{\xi\geq 0}$$

$$+ \left\{\frac{u-\eta}{1-\eta}1_{u\geq\eta} + \frac{\eta-u}{\eta}1_{u<\eta}\right\} 1_{\xi<0}.$$

If ν is a Lévy measure on $\mathbb{R}^2$, satisfying $\int(|x| \wedge 1)\nu(dx) < \infty$ with marginal tail integrals U_1, U_2 and Lévy copula F of Example 3.1, the Lévy process
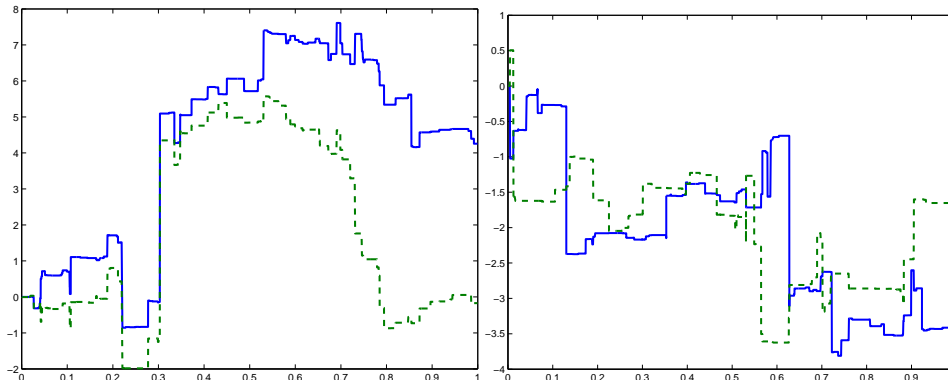
Figure 1: Trajectories of two variance gamma processes with dependence structure given by the Lévy copula of Example 3.1. In both graphs both variance gamma processes are driftless and have parameters $c = 10$, $\lambda_- = 1$ and $\lambda_+ = 1$ (cf. Equation (2.4)). In the left graph, the dependence between the two components is strong both in terms of sign and absolute value ($\eta = 0.9$ and $\theta = 3$): the processes jump mostly in the same direction and the sizes of jumps are similar. In the right graph the dependence of absolute values is weak ($\theta = 0.5$) and the dependence of jump signs is negative ($\eta = 0.25$).

with characteristic function (4.10) can be simulated as follows. Let $\{V_i\}$ and $\{\Gamma_i^1\}$ be as in Theorem 4.3 and let $\{W_i\}$ be an independent sequence of independent random variables, uniformly distributed on $[0, 1]$. For each i, let $\Gamma_i^2 = F_{\Gamma_i^1}^{-1}(W_i)$. Then the Lévy process that we want to simulate is given by Equation (4.9).

Figure 1 shows the simulated trajectories of two variance gamma processes with dependence structure given by the Lévy copula of Example 3.1 with different values of parameters. The number of jumps for each trajectory was limited to 2000 and the inverse tail integral of the variance gamma Lévy measure was computed by inverting numerically the exponential integral function (function `expint` available in MATLAB). Simulating two trajectories with 2000 jumps each takes about 1 second on a Pentium III computer running MATLAB, but this time could be reduced by several orders of magnitude if the inverse exponential integral function is tabulated and a lower-level programming language (e.g. C++) is used.

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5 Pricing multi-asset options using Lévy copulas

In this section we present a case study showing how one particular model, constructed using Lévy copulas, can be used to price multi-asset options.

The model   We suppose that under the risk-neutral probability, the prices $(S_t^1)_{t\geq 0}$ and $(S_t^2)_{t\geq 0}$ of two risky assets satisfy

$$S_t^1 = e^{rt+X_t^1}, \qquad S_t^2 = e^{rt+X_t^2}, \tag{5.1}$$

where (X^1, X^2) is a Lévy process on $\mathbb{R}^2$ with characteristic triplet $(0, \nu, b)$ with respect to zero truncation function. X^1 and X^2 are supposed to be variance gamma processes, that is, the margins ν_1 and ν_2 of ν are of the form (2.4) with parameters $c^1, \lambda_+^1, \lambda_-^1$ and $c^2, \lambda_+^2, \lambda_-^2$. The Lévy copula F of ν is supposed to be of the form (3.1) with parameters θ and η. The no-arbitrage condition imposes that for $i = 1, 2$, $\lambda_+^i > 1$ and the drift coefficients satisfy

$$b_i = c_i \log\left(1 - \frac{1}{\lambda_+^i} + \frac{1}{\lambda_-^i} - \frac{1}{\lambda_+^i \lambda_-^i}\right).$$

The problem   In the rest of this section, model (5.1) will be used to price two different kinds of multi-asset options: the option on weighted average, whose payoff at expiration date T is given by

$$H_T = \left(\sum_{i=1}^{2} w_i S_T^i - K\right)^+ \quad \text{with} \quad w_{1,2} \geq 0 \quad \text{and} \quad w_1 + w_2 = 1,$$

and the best-of or alternative option with payoff structure

$$H_T = \left(N \max\left(\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}\right) - K\right)^+$$

Option pricing by Monte Carlo   Basket options, described above can be priced by Monte Carlo method using European options on individual stocks as control variates. Denote the discounted payoffs of European options by

$$V_T^i = e^{-rT}(S_T^i - K)^+ \quad \text{for } i = 1, 2.$$

and the discounted payoff of the basket option by $V_T = e^{-rT} H_T$. Then the Monte Carlo estimate of basket option price is given by

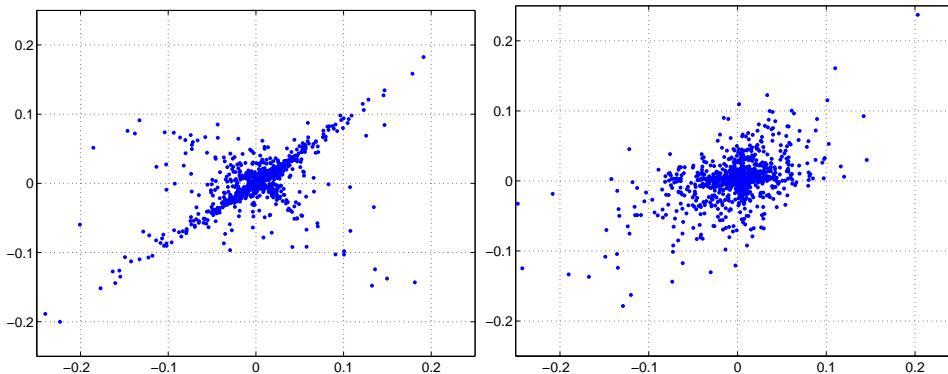$$\widehat{E[V_T]} = \bar{V}_T + a_1(E[V_T^1] - \bar{V}_T^1) + a_2(E[V_T^2] - \bar{V}_T^2),$$

Figure 2: Scatter plots of returns in a 2-dimensional variance gamma model with correlation $\rho = 50\%$ and different tail dependence. Left: strong tail dependence ($\eta = 0.75$ and $\theta = 10$). Right: weak tail dependence ($\eta = 0.99$ and $\theta = 0.61$).

where a bar over a random variable denotes the sample mean over N i.i.d. realizations of this variable, that is, $\bar{V}_T = \frac{1}{N}\sum_{i=1}^{N} V_T^{(i)}$, where $V_T^{(i)}$ are independent and have the same law as V_T. The coefficients a_1 and a_2 should be chosen in order to minimize the variance of $\widehat{E[V_T]}$. It is easy to see that this variance is minimal if $a = \Sigma a^0$, where $\Sigma_{ij} = \mathrm{Cov}(V_T^i, V_T^j)$ and $a_i^0 = \mathrm{Cov}(V_T, V_T^i)$. In practice these covariances are replaced by their in-sample estimates; this may introduce a bias into the estimator $\widehat{E[V_T]}$, but for sufficiently large samples this bias is small compared to the Monte Carlo error [6].

To illustrate the option pricing procedure, we fixed the following parameters of the marginal distributions of the two assets: $c^1 = c^2 = 25$, $\lambda_+^1 = 28.9$, $\lambda_-^1 = 21.45$, $\lambda_+^2 = 31.66$ and $\lambda_-^2 = 25.26$. In the parametrization (2.3) this corresponds to $\theta^1 = \theta^2 = -0.2$, $\kappa^1 = \kappa^2 = 0.04$, $\sigma^1 = 0.3$ and $\sigma^2 = 0.25$. To emphasize the importance of tail dependence for pricing multi-asset options, we used two sets of dependence parameters, which correspond both to a correlation of 50% (the correlation is computed numerically) but lead to returns with very different tail dependence structures:

Pattern 1 Strong tail dependence: $\theta = 10$ and $\eta = 0.75$. The scatter plot of returns is shown in Figure 2, left graph. Although the signs of returns may be different, the probability that the returns will be large in absolute value simultaneously in both components is very high.
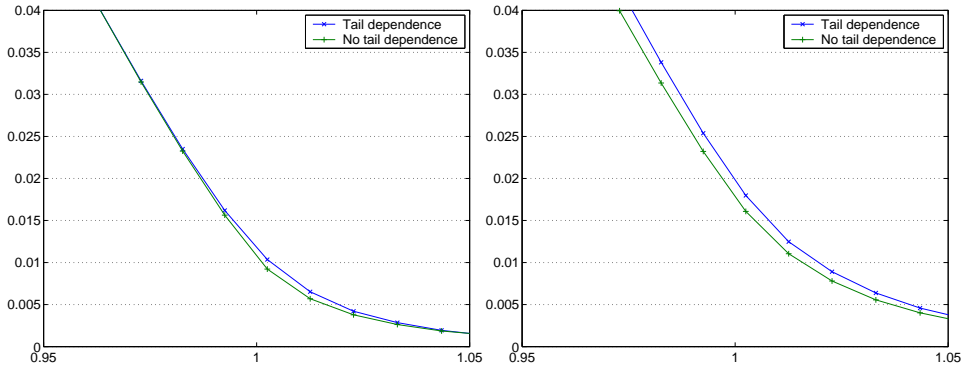
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Figure 3: Prices of options on weighted average (left) and of best-of options (right) for two different dependence patterns.

Pattern 2 Weak tail dependence: $\theta = 0.61$ and $\eta = 0.99$. The scatter plot of returns in shown in Figure 2, right graph. With this dependence structure the returns typically have the same sign but their absolute values are only weakly correlated.

In each of the two cases, a sample of 1000 realizations of the couple (X_T^1, X_T^2) with $T = 0.02$ (one-week options) was simulated using the procedure described in Example 4.1. The cutoff parameter τ (see Equation (4.11)) was taken equal to 1000, which lead to limiting the average number of jumps for each trajectory to about 40. For this value of τ, $U_i^{-1}(\tau)$ is of order of 10^{-19} for both assets. Since for the variance gamma model the convergence of U^{-1} to zero as $\tau \to \infty$ is exponential, the error resulting from the truncation of small jumps is of the same order, hence, negligible.

Figure 3 shows the prices of basket options, computed for different strikes with dependence patterns given above. The initial asset prices were $S_0^1 = S_0^2 = 1$, and the interest rate was taken to be $r = 0.03$. For the option on weighted average, the weights w_i were both equal to 0.5 and for the best-of option the coefficient was $N = 1$. The prices of European options, used for variance reduction, were computed using the Fourier transform algorithm (see [17] for the detailed description of our procedure and [3] for the original reference). The standard deviation of Monte Carlo estimates of option prices was below $2 \cdot 10^{-4}$ at the money in all cases.

The difference between option prices computed with and without tail dependence is clearly important for both types of options: as seen from Figure 3, neglecting tail dependence may easily lead to a 10% error on the option price at the money. On the other hand, this example shows that

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using Lévy copulas allows to take into account the tail dependence and discriminate between two situations that would be undistinguishable in a log-normal framework.

Acknowledgement

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