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CHAPTER 10

Risk Neutral Pricing of Counterparty Risk *

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Contents

1 Introduction

"Failure is not an option"

NASA "Apollo XIII" mission rescue motto

In this chapter we show how to handle counterparty risk when pricing some basic financial products. In particular we are analyzing in detail *counterparty-risk* (or Default-risk) *Interest Rate Swaps* and *counterparty-risk* equity return swaps. The reason to introduce counterparty risk when evaluating a contract is linked to the fact that many financial contracts are traded over the counter (OTC), so that the credit quality of the counterparty can be important. This is particularly appropriated when thinking of the different defaults experienced by some important companies during the last years. Also, regulatory issues related to the Basel II framework encourage the inclusion of counterparty risk into valuation.

We face the problem from the viewpoint of a safe (default-free) counterparty entering a financial contract with another counterparty that has a positive probability of defaulting before the maturity of the contract itself. We are assuming there are no guarantees in place (such as for example collateral). When investing in default risky assets we require a *risk premium* as a reward for assuming the default risk. If we think, for example, of a corporate bond, we know that the yield is higher than the corresponding yield of an equivalent treasury bond, and this difference is usually called *credit spread*. The (positive) credit spread implies a lower price for the bond when compared to default free bonds. This is a typical feature of every asset: The value of a generic claim traded with a counterparty subject to default risk is always smaller than the value of the same claim traded with a counterparty having a null default probability.

In the paper we focus on the following points in particular:

- We assume absence of guarantees such as collateral;
- Illustrate how the inclusion of counterparty risk in the valuation can make a payoff model dependent by adding one level of optionality;
- Use the risk neutral default probability for the counterparty by extracting it from Credit Default Swap (CDS) data;
- Because of the previous point, the chosen default model will have to be calibrated to CDS data;
- When possible (and we will do so in Part III), take into account the correlation between the underlying of the contract and the default of the counterparty.

More in detail, when evaluating default risky assets, one has to introduce the default probabilities in the pricing models. We consider Credit Default Swaps as liquid sources of market risk-neutral default probabilities. Different models can be used to calibrate CDS data and obtain default probabilities: In Brigo and Tarenghi (2004, 2005) for example *firm value models (or structural models)* are used, whereas in Brigo and Alfonsi (2005) and Brigo and Cousot (2004) a stochastic intensity model is used. In this chapter when dealing with the interest rate swap examples, we strip default probabilities from CDS data in a model independent way that assumes independence between the default time and interest rates.

When moving to the equity return swap example, we resort instead to the firm value models of Brigo and Tarenghi (2004, 2005), that can be calibrated again to CDS data in an exact way.

The Chapter starts in Part I with a general formula for counterparty risk valuation in a derivative transaction. We show that the derivative price in presence of counterparty risk is just the default free price minus a discounted option term in scenarios of early default times the loss given default (also called "expected loss"). The option is on the residual present value at time of default. We notice that even payoffs whose valuation is model independent become model dependent due to counterparty risk. This aspect is rather dramatic when trying to incorporate counterparty risk in a way that does not destroy the default-free valuation models.

We apply the general result to two fundamental areas in Parts II and III of the chapter.

In Part II we compute counterparty risk for portfolios of interest rate swaps, possibly in presence of netting agreements. We will derive quick approximated formulas and test them against full Monte Carlo simulation of the price. The derivation will assume independence between interest rates and the default time and will be model independent. The framework is also suited to computing counterparty risk on non-standard swap contracts such as zero coupon swaps, amortizing swaps etc.

In Part III we compute the counterparty risk price of an equity payoff. We focus on equity return swaps as a fundamental example that can be easily generalized to other payoffs. We resort to a structural model because this allows us to take into account the correlation between the underlying equity and the counterparty default in a natural way. We illustrate how the correlation has an impact on the valuation of counterparty risk. Furthermore, contrary to traditional structural models, our first time passage models will be incorporating CDS data in an exact way, following Brigo and Tarenghi (2004, 2005). Conclusions close the chapter.

Part I

General Valuation of Counterparty Risk

2 The Probabilistic Framework

This section contains our probabilistic assumptions.

We place ourselves in a probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$. The usual interpretation of this space as an experiment can help intuition. The generic experiment result is denoted by $\omega \in \Omega$; Ω represents the set of all possible outcomes of the random experiment, and the σ-field $\mathcal{G}$ represents the set of events $A \subset \Omega$ with which we shall work. The σ-field $\mathcal{G}_t$ represents the information available up to time t. We have $\mathcal{G}_t \subseteq \mathcal{G}_u \subseteq \mathcal{G}$ for all $t \leq u$, meaning that "the information increases in time", never exceeding the whole set of events $\mathcal{G}$. The family of σ-fields $(\mathcal{G}_t)_{t\geq 0}$ is called filtration.

If the experiment result is ω and $\omega \in A \in \mathcal{G}$, we say that the event A occurred. If $\omega \in A \in \mathcal{G}_t$, we say that the event A occurred at a time smaller or equal to t.

We use the symbol $\mathbb{E}$ to denote expectation.

The default time τ will be defined on this probability space.

This space is endowed with a right-continuous and complete sub-filtration $\mathcal{F}_t$ representing all the observable market quantities but the default event (hence $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(\{\tau \leq u\} : u \leq t)$ is the right-continuous filtration generated by the default event). We set $\mathbb{E}_t(\cdot) := \mathbb{E}(\cdot|\mathcal{G}_t)$.

In more colloquial terms, throughout the chapter $\mathcal{G}_t$ is the filtration modeling the market information up to time t, including explicit default monitoring up to t, whereas $\mathcal{F}_t$ is the default-free market information up to t (FX, interest rates etc), without default monitoring.

In the first application of the paper, given in Part II, we assume independence between τ and interest rates, but we make no specific assumption on the model for τ. An example of such a model could be an intensity model with deterministic intensity or with stochastic intensity independent of interest rates. However, the result holds in general with no need for a specific model. In the second application given in Part III we assume a first passage time structural model for τ.

In intensity models, in general $\mathcal{F}_t \subset \mathcal{G}_t$. In basic first passage time structural models, instead, as our AT1P model in Part III, we have $\mathcal{F}_t = \mathcal{G}_t$, i.e. all default information is already included in the default free market, and default is predictable in the probabilistic sense.

3 CDS as Sources of Implied Default Probabilities

We now recall briefly the CDS payoff and its risk neutral pricing formula in our probabilistic framework.

One of the most representative protection instruments that can be used against default is the Credit Default Swap (CDS). Consider two companies "A" (the *protection buyer*) and "B" (the *protection seller*) who agree on the following.

If a third reference company "C" (*the reference credit*) defaults at a time $\tau_C \in (T_a, T_b]$, "B" pays to "A" at time $\tau = \tau_C$ itself a certain "protection" amount L_{GD} (Loss Given the Default of "C"), supposed to be deterministic in the present paper. This amount is a *protection* for "A" in case "C" defaults. Typically L_{GD} is equal to a notional amount (usually set to 1) minus a recovery rate "R_{EC}", $\text{L}_{\text{GD}} = 1 - \text{R}_{\text{EC}}$.

In exchange for this protection, company "A" agrees to pay periodically to "B" a fixed "running" amount R, at a set of times $\{T_{a+1}, \dots, T_b\}$, $T_0 = 0$. These payments constitute the "premium leg" of the CDS (as opposed to the L_{GD} payment, which is termed the "protection leg"), and the rate R is fixed in advance at time 0; the premium payments go on up to default time τ if this occurs before maturity T_b, or until maturity T_b if no default occurs.

"B"	$\rightarrow$	protection L_{GD} at default τ_C if $T_a < \tau_C \leq T_b$	$\rightarrow$	"A"
"B"	$\leftarrow$	rate R at $T_{a+1}, \dots, T_b$ or until default τ_C	$\leftarrow$	"A"

Formally, we may write the RCDS ("R" stands for running) payoff discounted value to time zero seen from "A" as

$$\Pi\text{RCDS}_{a,b} := -D(0,\tau)(\tau - T_{\beta(\tau)-1}) R \mathbf{1}_{\{T_a < \tau < T_b\}} - \sum_{i=a+1}^{b} D(0,T_i)\alpha_i R \mathbf{1}_{\{\tau \geq T_i\}} + \\ + \mathbf{1}_{\{T_a < \tau \leq T_b\}} D(0,\tau) \text{L}_{\text{GD}} \qquad (3.1)$$

where in general $t \in [T_{\beta(t)-1}, T_{\beta(t)})$, i.e. $T_{\beta(t)}$ is the first date among the T_i's that follows t, α_i is the year fraction between T_{i-1} and T_i, and $D(s,u)$ is the stochastic discount factor at time s for maturity u. Sometimes slightly different payoffs are considered for RCDS contracts, where the protection payment is postponed to the first time T_i following default. See for example Brigo (2005, 2005b) for a discussion on different payoffs and related implications.

We denote by $\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}})$ the price at time 0 of the above standard running CDS. In general, we can compute the CDS price according to the risk-neutral expectation $\mathbb{E}$ of the discounted payoff (see for example Bielecki and Rutkowski (2001), the risk neutral measure is denoted by $\mathbb{Q}$):

$$\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}) = \mathbb{E}\{\Pi_{\text{RCDS}_{a,b}}\}. \tag{3.2}$$

Let $P(0, T_i)$ be the price of a zero-coupon bond at 0 with maturity T_i. Under interest rates independent of the default time, straightforward computations lead to the price at initial time 0 as

$$\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}) = R \int_{T_a}^{T_b} P(0,t)(t - T_{\beta(t)-1}) d_t\mathbb{Q}(\tau > t) \tag{3.3}$$
$$-R \sum_{i=a+1}^{b} P(0, T_i)\alpha_i \mathbb{Q}(\tau \geq T_i) - \text{L}_{\text{GD}} \int_{T_a}^{T_b} P(0,t) d_t\mathbb{Q}(\tau > t)$$

so that if one has a formula for the curve of survival probabilities $t \mapsto \mathbb{Q}(\tau > t)$, one also has a formula for CDS.

A CDS is quoted through its "fair" R, in that the rate R that is quoted by the market at time 0 satisfies $\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}) = 0$. The fair rate R strongly depends on the default probabilities. The idea is to use quoted values of these fair R's with increasing maturities T_b (and initial resets all set to $T_a = 0$) to derive the default (survival) probabilities $\mathbb{Q}(\tau > t)$ assessed by the market, by inverting Formula (3.3).

4 The General Pricing Formula

Let us call T the final maturity of the payoff we are going to evaluate. If $\tau > T$ there is no default of the counterparty during the life of the product and the counterparty has no problems in repaying the investors. On the contrary, if $\tau \leq T$ the counterparty cannot fulfill its obligations and the following happens. At τ the Net Present Value (NPV) of the residual payoff until maturity is computed: If this NPV is negative (respectively positive) for the investor (defaulted counterparty), it is completely paid (received) by the investor (counterparty) itself. If the NPV is positive (negative) for the investor (counterparty), only a recovery fraction R_{EC} of the NPV is exchanged. Here all the expectations $\mathbb{E}_t$ are taken under the risk neutral measure $\mathbb{Q}$ and with respect to the filtration $\mathcal{G}_t$.

Let us call $\Pi^D(t)$ the payoff of a generic defaultable claim at t and $\text{C}_{\text{ASHFLOWS}}(u \div s)$ the net cash flows of the claim between time u and time s, discounted back at u, all payoffs seen from the point of view of the company facing counterparty risk. Then we have $\text{NPV}(\tau) = \mathbb{E}_\tau\{\text{C}_{\text{ASHFLOWS}}(\tau, T)\}$ and

$$\begin{aligned}\Pi^D(t) \;=\;& \mathbf{1}_{\{\tau > T\}} \text{C}_{\text{ASHFLOWS}}(t,T) + \\ & \mathbf{1}_{\{t < \tau \leq T\}} \left[\text{C}_{\text{ASHFLOWS}}(t,\tau) + D(t,\tau) \left(\text{R}_{\text{EC}} \left(\text{NPV}(\tau)\right)^+ - \left(-\text{NPV}(\tau)\right)^+ \right) \right].\end{aligned} \tag{4.1}$$

This last expression is the general price of the payoff under counterparty risk. Indeed, if there is no early default this expression reduces to risk neutral valuation of the payoff (first term in the right hand side); in case of early default, the payments due before default occurs are received (second term), and then if the residual net present value is positive only a recovery of it is received (third term), whereas if it is negative it is paid in full (fourth term).

Calling $\Pi(t)$ the payoff for an equivalent claim with a default-free counterparty, it is possible to prove the following

Proposition 4.1. (General counterparty risk pricing formula). *At valuation time t, and provided the counterparty has not defaulted before t, i.e. on $\{\tau > t\}$, the price of our payoff under counterparty risk is*

$$\boxed{\mathbb{E}_t\{\Pi^D(t)\} = \mathbb{E}_t\{\Pi(t)\} - L_{GD}\,\mathbb{E}_t\{\mathbf{1}_{\{t<\tau\le T\}} D(t,\tau)\,(NPV(\tau))^+\}} \tag{4.2}$$

where $L_{GD} = 1 - R_{EC}$ is the Loss Given Default *and the recovery fraction R_{EC} is assumed to be deterministic.*

We now prove the proposition.

Proof. Since

$$\Pi(t) = \text{Cashflows}(t,T) = \mathbf{1}_{\{\tau>T\}}\text{Cashflows}(t,T) + \mathbf{1}_{\{\tau\le T\}}\text{Cashflows}(t,T) \tag{4.3}$$

we can rewrite the terms inside the expectation in the right hand side of (4.2) as

$$\begin{aligned}
& \mathbf{1}_{\{\tau>T\}}\text{Cashflows}(t,T) + \mathbf{1}_{\{\tau\le T\}}\text{Cashflows}(t,T) \\
+ \; & \{(\text{R}_{\text{EC}} - 1)[\mathbf{1}_{\{\tau\le T\}} D(t,\tau)(\text{NPV}(\tau))^+]\} \\
= \; & \mathbf{1}_{\{\tau>T\}}\text{Cashflows}(t,T) + \mathbf{1}_{\{\tau\le T\}}\text{Cashflows}(t,T) \\
+ \; & \text{R}_{\text{EC}}\;\; \mathbf{1}_{\{\tau\le T\}} D(t,\tau)(\text{NPV}(\tau))^+ - \mathbf{1}_{\{\tau\le T\}} D(t,\tau)(\text{NPV}(\tau))^+
\end{aligned} \tag{4.4}$$

Conditional on the information at τ the second and the fourth terms are equal to

$$\begin{aligned}
& E_\tau[\mathbf{1}_{\{\tau\le T\}}\text{Cashflows}(t,T) - \mathbf{1}_{\{\tau\le T\}} D(t,\tau)(\text{NPV}(\tau))^+] \\
= \; & E_\tau[\mathbf{1}_{\{\tau\le T\}}[\text{Cashflows}(t,\tau) + D(t,\tau)\text{Cashflows}(\tau,T) - D(t,\tau)(E_\tau[\text{Cashflows}(\tau,T)])^+]] \\
= \; & \mathbf{1}_{\{\tau\le T\}}[\text{Cashflows}(t,\tau) + D(t,\tau)E_\tau[\text{Cashflows}(\tau,T)] - D(t,\tau)(E_\tau[\text{Cashflows}(\tau,T)])^+] \\
= \; & \mathbf{1}_{\{\tau\le T\}}[\text{Cashflows}(t,\tau) - D(t,\tau)(E_\tau[\text{Cashflows}(\tau,T)])^-] \\
= \; & \mathbf{1}_{\{\tau\le T\}}[\text{Cashflows}(t,\tau) - D(t,\tau)(E_\tau[-\text{Cashflows}(\tau,T)])^+] \\
= \; & \mathbf{1}_{\{\tau\le T\}}[\text{Cashflows}(t,\tau) - D(t,\tau)(-\text{NPV}(\tau))^+]
\end{aligned} \tag{4.5}$$

since trivially

$$\mathbf{1}_{\{\tau\le T\}}\text{Cashflows}(t,T) = \mathbf{1}_{\{\tau\le T\}}\{\text{Cashflows}(t,\tau) + D(t,\tau)\text{Cashflows}(\tau,T)\} \tag{4.6}$$

and $f = f^+ - f^- = f^+ - (-f)^+$.

Then we can see that after conditioning on the information at time τ (4.4) and substituting the second and the fourth terms for (4.5), the expected value of (4.1) with respect to $\mathcal{F}_t$ coincides exactly with (4.2) by the properties of iterated expectations. □

Remark 4.2. *(***Counterparty risk as an option and induced model dependence)**
It is clear that the value of a defaultable claim is the sum of the value of the corresponding default-free claim minus an option part, in the specific a call option (with zero strike) on the residual NPV giving nonzero contribution only in scenarios where $\tau \leq T$. Counterparty risk thus adds an optionality level to the original payoff. This renders the counterparty risky payoff model dependent even when the original payoff is model independent. This implies, for example, that while the valuation of swaps without counterparty risk is model independent, requiring no dynamical model for the term structure (no volatility and correlations in particular), the valuation of swaps under counterparty risk will require an interest rate model. This implies that quick fixes of existing pricing routines to include counterparty risk are difficult to obtain.

Part II

Interest rate swaps portfolios

In this first part devoted to specific products we deal with counterparty risk in interest rate swaps (IRS) portfolios. The results can easily be transferred to single IRS with nonstandard features such as zero coupon IRS, amortizing IRS, bullet IRS etc. It suffices to suitably define the α and χ portfolio coefficients below.

This part is structured as follows.

In Section 5 we apply the general formula to a single IRS. We find the already known result (see also Sorensen and Bollier(1994), Arvanitis and Gregory (2001) Chapter 6, Bielecki and Rutkowski (2001) Chapter 14, Cherubini (2005), or Brigo and Mercurio (2005), among other references) that the component of the IRS price due to counterparty risk is the sum of swaption prices with different maturities, each weighted with the probability of defaulting around that maturity. Things become more interesting when we consider a portfolio of IRS's towards a single counterparty in presence of a netting agreement. When default occurs, we need to consider the option on the residual present value of the whole portfolio. This option cannot be valued as a standard swaption, and we need either to resort to Monte Carlo simulation (under the LIBOR model, or alternatively the swap model) or to derive analytical approximations. We derive an analytical approximation based on the standard "drift freezing" technique for swaptions pricing in the LIBOR model (Brace Gatarek and Musiela (1997), see also Rebonato (1998), among many other references; Chapter 6 of Brigo and Mercurio (2001) presents a standard derivation of the formula and numerical tests). We mimic the basic drift freezing procedure for implementation simplicity, although several improvements have been presented in the recent years for standard swaptions (see for example Jackel and Rebonato (2003) and Rebonato (2003)). The reader can easily mimic this more advanced approximation following the derivation presented here.

We find a formula that we can expect to work in case all IRS in the portfolio have the same direction: If we have a portfolio of IRS that are all long or short towards the same counterparty, the pricing problems become similar to that of a swap with different multiples of LIBOR and strikes at each payment date. The problem is then reduced to pricing an option on an IRS with non-standard flows. We do so by means of the above-mentioned drift freezing technique and derive an analytical approximation. We test this approximation using Monte Carlo simulation under different stylized compositions of the netting portfolio.

Consistently with well known results for standard swaptions under the LIBOR model with drift freezing, we find the approximation to work well within reasonable limits.

The more interesting part, however, is allowing for IRS's towards a given counterparty but going into both directions, long and short. This time the situation becomes more complicated: we can see the residual present value at the early default time as an option on the difference of two swap rates, each approximately lognormal. We study the drift freezing procedure and also an alternative three moments matching procedure. We test both approximations against Monte Carlo simulation under different portfolio configurations. We obtain a good approximation in most situation.

The approximated formula is well suited to risk management, where the computational time under each risk factors scenario is crucial and an analytical approximation may be needed to contain it.

5 Counterparty Risk in single Interest Rate Swaps (IRS)

For the theory relative to the Interest Rate Swap we refer for example to Brigo and Mercurio (2001). Let us suppose that we are a default-free counterparty "A" entering a payer swap with a defaultable counterparty "B", exchanging fixed for floating payments at times $T_{a+1}, \dots, T_b$.

Denote by β_i the year fraction between T_{i-1} and T_i, and by $P(t, T_i)$ the default free zero coupon bond price at time t for maturity T_i. We take a unit notional on the swap. The contract requires us to pay a fixed rate K and to receive the floating rate L resetting one-period earlier until the default time τ of "B" or until final maturity T if $\tau > T$. The fair (forward-swap) rate K at a given time t in a default-free market is the one which renders the swap zero-valued in t.

In the risk-free case the discounted payoff for a payer IRS is

$$\sum_{i=a+1}^{b} D(t,T_i)\ \beta_i\ (L(T_{i-1},T_i) - K) \tag{5.1}$$

and the forward swap rate rendering the contract fair is

$$K = S(t;T_a,T_b) = S_{a,b}(t) = \frac{P(t,T_a) - P(t,T_b)}{\sum_{i=a+1}^{b} \beta_i P(t,T_i)}. \tag{5.2}$$

Of course, if we consider the possibility that "B" may default, the correct spread to be paid in the fixed leg is lower, as we are willing to be rewarded for bearing this default risk. In particular, using the previous formula (4.2) we find

$$\mathrm{IRS}^D(t) = \mathrm{IRS}(t) - \mathrm{EL}(t) \tag{5.3}$$

where $\mathrm{EL}(\cdot)$ is the expected loss due to default. EL can be computed as follows under independence of the default time τ from interest rates:

$$\begin{aligned} \mathrm{EL}(t) &= \mathrm{L{\scriptstyle GD}}\ \mathbb{E}_t\{\mathbf{1}_{\{\tau \le T_b\}} D(t,\tau)(NPV(\tau))^+\} = \\ &= \mathrm{L{\scriptstyle GD}} \int_{T_a}^{T_b} \mathrm{S{\scriptstyle WAPTION}}(t; s, T_b, K, S(t;s,T_b), \sigma_{s,T_b}) d_s \mathbb{Q}(\tau \le s) \end{aligned} \tag{5.4}$$

being $\text{Swaption}(t; s, T_b, K, S(t; s, T_b), \sigma_{s,T_b})$ the price in t of a swaption with maturity s, strike K, underlying forward swap rate $S(t; s, T_b)$, volatility σ_{s,T_b} and underlying swap with final maturity T_b. When $s = T_j$ for some j we replace the arguments s, T_b by indices j, b.

The proof is easy: given independence between τ and interest rates, and given that the residual NPV is a forward start IRS starting at the default time, the option on the residual NPV is a sum of swaptions with maturities ranging the possible values of the default time, each weighted (thanks to independence) by the probabilities of defaulting around each time value.

We can simplify (5.4) through some assumptions: We allow the default to happen only at points T_i of the grid of payments of the fixed leg. In particular two different specifications could be applied: One for which the default is anticipated to the first T_i preceding τ and one for which it is postponed to the first T_i following τ. In this way the expected loss in (5.4) is simplified. Indeed, in the case of the postponed (P) payoff we obtain

$$\begin{aligned} \text{EL}^P(t) &= \text{Lgd} \sum_{i=a+1}^{b-1} \mathbb{Q}\{\tau \in (T_{i-1}, T_i]\} \, \text{Swaption}_{i,b}(t; K, S_{i,b}(t), \sigma_{i,b}) \\ &= \text{Lgd} \sum_{i=a+1}^{b-1} (\mathbb{Q}(\tau > T_{i-1}) - \mathbb{Q}(\tau > T_i)) \, \text{Swaption}_{i,b}(t; K, S_{i,b}(t), \sigma_{i,b}) \end{aligned} \tag{5.5}$$

and this can be easily computed summing across the T_i's and using the default probabilities implicitly given in market CDS prices.
A similar result can be obtained considering the anticipated (A) default

$$\begin{aligned} \text{EL}^A(t) &= \text{Lgd} \sum_{i=a+1}^{b} \mathbb{Q}\{\tau \in (T_{i-1}, T_i]\} \, \text{Swaption}_{i-1,b}(t; K, S_{i-1,b}(t), \sigma_{i-1,b}) \\ &= \text{Lgd} \sum_{i=a+1}^{b} (\mathbb{Q}(\tau > T_{i-1}) - \mathbb{Q}(\tau > T_i)) \, \text{Swaption}_{i-1,b}(t; K, S_{i-1,b}(t), \sigma_{i-1,b}) \end{aligned} \tag{5.6}$$

We carried out some numerical experiments to analyze the impact of postponement or anticipation on counterparty risk based on data of March 10th, 2004. Results are given in Brigo and Masetti (2005) and show that the postponement or anticipation of the default time is not affecting the counterparty risk price in any important way.

6 Counterparty Risk in a Portfolio of IRS with netting

In case we are dealing with a portfolio of IRS's towards a single counterparty under a netting agreement, we need to take into account the netting possibilities. This complicates matters considerably, as we are going to see shortly. We will derive an analytical approximation tested by different netting coefficients.

Remark 6.1 (Portfolio of IRS's). *In a portfolio of IRS's, consisting of several single IRS towards the same counterparty with different tenors and maturities put together, some long and some short, we may think of assembling the cash flows at each resetting date. Floating rates add up and subtract into multiples (positive or negative) of LIBOR rates at each reset and the fixed rates (strikes) of the basic IRS's behave similarly.*

Suppose that we have a portfolio of N IRS's with homogeneous resetting dates but different maturities and inception dates. Let

$$\alpha_i := \beta_i|\sum_{j=1}^{N} A_i^j \phi_j|, \quad K_i := \beta_i|\sum_{j=1}^{N} A_i^j K_i^j \phi_j| \tag{6.1}$$

$$\chi_i := \text{sign}(\sum_{j=1}^{N} A_i^j \phi_j), \quad \psi_i := \text{sign}(\sum_{j=1}^{N} A_i^j K_i^j \phi_j) \tag{6.2}$$

for all $i \in [a+1, b]$, where: $A_i^j \geq 0$ is the notional amount relative to the j-th IRS on the resetting date T_i (this allows for inclusion of any amortizing plan); ϕ_j is the payer/receiver fixed rate flag which takes values in $\{-1, 1\}$ (e.g. 1 for payer, -1 for receiver); $K_i^j > 0$ is the fixed rate of the j-th IRS for the T_i reset.

Example 1. *We show here an example of how the χ_i may be different from the ψ_i. Fix a reset T_i and consider a portfolio with three IRS, having the same notional amount (suppose $A_i^j = 1$ for all $j \in [1, 3]$). Suppose that we are facing the following structure:*

- *IRS $j = 1$ (Payer fixed rate): $K_i^1 = 1\%$;*
- *IRS $j = 2$ (Payer fixed rate): $K_i^2 = 2\%$;*
- *IRS $j = 3$ (Receiver fixed rate): $K_i^3 = 4\%$.*

It follows that $\chi_i = 1$ whereas $\psi_i = -1$.

We denote by $L(T_{i-1}, T_i)$ the Libor rate on the resetting period T_{i-1} and T_i (where T_i is expressed in terms of year-fraction).

The total portfolio discounted payoff at time $t \leq T_a$ may be written as

$$\Pi_{\text{Pirs}}(t) = \sum_{i=a+1}^{b} D(t, T_i)[\chi_i \alpha_i L(T_{i-1}, T_i) - \psi_i K_i] \tag{6.3}$$

$$= \sum_{i=a+1}^{b} D(t, T_i)\chi_i[\alpha_i L(T_{i-1}, T_i) - \widetilde{K}_i] \tag{6.4}$$

where $\widetilde{K}_i := (\frac{\psi_i}{\chi_i})K_i$.
The α_i is the positive total year fraction (also called netting coefficient) in front of the LIBOR rates in the total portfolio of IRS towards a given counterparty.
This framework can also be used for single non-standard IRS (zero coupon, bullet, amortizing...) by suitably defining the α's and K's.
The $\widetilde{K}_i$ represents the cumulated fixed rate of the total portfolio to be exchanged at time T_i, when the valuation is made at time t.

The expected value at time t for a default-free portfolio is known to be

$$\mathbb{E}_t[\Pi_{\text{Pirs}}(t)] = \sum_{i=a+1}^{b} P(t, T_i)\chi_i[\alpha_i F_i(t) - \widetilde{K}_i] \tag{6.5}$$

where each expectation in the sum has been easily computed by resorting for example to the related forward measure, and $F_i(t)$ is the forward LIBOR rate at time t for expiry at T_{i-1} and maturity T_i. This expected value represents the swap price without counterparty risk, and we see that this price is model independent. One only needs the initial time-t interest rate curve to compute forward rates $F_i(t)$ and discounts $P(t,T_i)$, with no need to postulate a dynamics for the term structure. Using formula (4.2) we can compute the expected value for the IRS portfolio under counterparty risk by

$$\mathbb{E}_t[\Pi^D_{\text{Pirs}}(t)] = \mathbb{E}_t[\Pi_{\text{Pirs}}(t)] - \text{L}_{\text{GD}}\ \mathbb{E}_t[\mathbf{1}_{\{t<\tau\le T_b\}} D(t,\tau)(\text{NPV}(\tau))^+] \tag{6.6}$$

where $\text{NPV}(\tau) = \mathbb{E}_\tau[\Pi_{\text{Pirs}}(\tau)]$.
The expected loss (EL) of (6.6) can be rewritten as

$$\begin{aligned}\text{EL}(t) &:= \text{L}_{\text{GD}}\ \mathbb{E}_t[\mathbf{1}_{\{\tau\le T_b\}} D(t,\tau)(\text{NPV}(\tau))^+] \\ &= \text{L}_{\text{GD}}\ \mathbb{E}_t[\sum_{i=a+1}^{b} \mathbf{1}_{\{\tau\in(T_{i-1},T_i]\}} D(t,\tau)(\text{NPV}(\tau))^+] \\ &= \text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{E}_t[\mathbf{1}_{\{\tau\in(T_{i-1},T_i]\}} D(t,\tau)(\text{NPV}(\tau))^+] \end{aligned} \tag{6.7}$$

Since we are assuming independence of τ from interest rates, if we postpone the default event up to the first T_i following τ, i.e.

$$\inf\{T_i :\ i \in \mathbb{Z},\ T_i\ \ge \tau\} \tag{6.8}$$

we finally have that

$$\text{EL}(t) = \text{L}_{\text{GD}} \sum_{i=a+1}^{b} \mathbb{Q}_{\{\tau\in(T_{i-1},T_i]\}} \mathbb{E}_t[D(t,T_i)(\text{NPV}(T_i))^+]. \tag{6.9}$$

Recall that in this case

$$\text{NPV}(T_i) = \sum_{k=i+1}^{b} P(T_i,T_k)\chi_k[\alpha_k F_k(T_i) - \widetilde{K}_k]$$

Since the counterparty-risky portfolio is decomposed into a swap (with non-standard coefficients) and a weighted sum of expectations on $\text{NPV}(\tau)$'s, the only issue we are facing now is to get an evaluation of $\mathbb{E}_t[D(t,T_i)(\text{NPV}(T_i))^+]$.
After multiplying and dividing by $\widehat{C}_{i,b}(T_i) := \sum_{h=i+1}^{b} \alpha_h P(T_i,T_h)$, this expectation can be rewritten as

$$\mathbb{E}_t[D(t,T_i)\widehat{C}_{i,b}(T_i)(\widehat{S}_{i,b}(T_i) - \widehat{K}(T_i))^+] \tag{6.10}$$

where, if we set for all T, $\widehat{w}_k(T) := \alpha_k P(T,T_k)/\widehat{C}_{i,b}(T)$ we have

$$\widehat{S}_{i,b}(T) := \sum_{k=i+1}^{b} \widehat{w}_k(T)\chi_k F_k(T), \quad \widehat{K}(T) := \sum_{k=i+1}^{b} \widehat{w}_k(T)\chi_k \frac{\widetilde{K}_k}{\alpha_k} \tag{6.11}$$

6.1 Approximating the variance under lognormality assumptions: The drift freezing approximation

Now consider the following approximation:

$$\widehat{S}_{i,b}(T_i) \approx \sum_{k=i+1}^{b} \widehat{w}_k(t)\chi_k F_k(T_i)$$

so that $d\widehat{S}_{i,b}(t') \approx \sum_{k=i+1}^{b} \widehat{w}_k(t)\chi_k dF_k(t')$, for $t' \in [t, T_i]$.
It follows by arguments completely analogous to those used for the approximated swaption pricing formula (Brace, Gatarek and Musiela (1997), Rebonato (1998), and Brigo and Mercurio (2001), Proposition 6.13.1) that the variance of $\widehat{S}_{i,b}(T_i)$ at time t can be easily approximated by

$$\nu_{i,b}^2 = \nu_{i,b}^2(t,T_i) \approx \widehat{S}_{i,b}(t)^{-2} \sum_{h,k=i+1}^{b} \widehat{w}_h(t)\widehat{w}_k(t)\chi_h\chi_k F_h(t)F_k(t)\rho_{h,k} \int_t^{T_i} \sigma_h(s)\sigma_k(s)ds \quad (6.12)$$

where σ_h and σ_k are the instantaneous volatilities of the forward rates F_h and F_k whereas $\rho_{h,k}$ is the instantaneous correlation between the Brownian motions of F_h and F_k. Notice that this procedure is very close to a two moment matching technique. We investigate a three moment matching technique in the next section.

Finally, changing numeraire and using the approximated dynamics, if $\widehat{S}_{i,b}(t)$ and $\widehat{K}$ have the same sign

$$\begin{aligned}
\mathbb{E}_t[D(t,T_i)(\mathrm{NPV}(T_i))^+] &= \mathbb{E}_t^B[B(t)(\mathrm{NPV}(T_i))^+/B(T_i)] \\
&= \widehat{\mathbb{E}}_t^{i,b}[\widehat{C}_{i,b}(t)(\mathrm{NPV}(T_i))^+/\widehat{C}_{i,b}(T_i)] \\
&= \widehat{C}_{i,b}(t)\widehat{\mathbb{E}}_t^{i,b}[(\widehat{S}_{i,b}(T_i) - \widehat{K}(T_i))^+] \\
&\approx \widehat{C}_{i,b}(t)\mathrm{Black}(\widehat{S}_{i,b}(t), \nu_{i,b}^2, \widehat{K})
\end{aligned} \quad (6.13)$$

where

$$\mathrm{Black}(\widehat{S}_{i,b}(t), \nu_{i,b}^2, \widehat{K}) = \widehat{S}_{i,b}(t)N(d1) - \widehat{K}N(d2)$$

with

$$d1 = \frac{\phi \ln(\frac{\widehat{S}_{i,b}(t)}{\widehat{K}}) + \phi\frac{1}{2}\nu_{i,b}^2}{\nu_{i,b}}$$

$$d2 = \frac{\phi \ln(\frac{\widehat{S}_{i,b}(t)}{\widehat{K}}) - \phi\frac{1}{2}\nu_{i,b}^2}{\nu_{i,b}}$$

$$\phi := \begin{cases} +1, & \text{if } \widehat{S}_{i,b}(t) > 0 \text{ and } \widehat{K} > 0; \\ -1, & \text{if } \widehat{S}_{i,b}(t) < 0 \text{ and } \widehat{K} < 0; \end{cases}$$

and $N(d)$ being the cumulative standard normal distribution function.

If instead $\widehat{S}_{i,b}(t) > 0$ and $\widehat{K} < 0$, the price is simply reduced to a forward on $\widehat{S}_{i,b}(t)$ whereas for $\widehat{S}_{i,b}(t) < 0$ and $\widehat{K} > 0$ the price is zero.

Notice that $\widehat{S}_{i,b}$ is a martingale under the measure associated with the numeraire $\widehat{C}_{i,b}$ since it can be written as a portfolio of zero coupon bonds divided by the numeraire itself. Indeed, by definition of $\widehat{S}$ we can write

$$\begin{aligned}
\widehat{S}_{i,b}(t') &= \sum_{k=i+1}^{b} \widehat{w}_k(t')\chi_k F_k(t') \\
&= \sum_{k=i+1}^{b} \frac{\alpha_k P(t',T_k)}{\widehat{C}_{i,b}(t')}\chi_k F_k(t') \\
&= \sum_{k=i+1}^{b} \frac{\alpha_k P(t',T_k)}{\widehat{C}_{i,b}(t')}\chi_k \frac{1}{\beta_k}\left(\frac{P(t',T_{k-1})}{P(t',T_k)} - 1\right) \\
&= \sum_{k=i+1}^{b} \frac{\alpha_k}{\widehat{C}_{i,b}(t')}\chi_k \frac{1}{\beta_k}(P(t',T_{k-1}) - P(t',T_k)) \qquad (6.14)
\end{aligned}$$

The above pricing formula has to be handled carefully. Notice in particular that the initial condition of the approximated dynamics, i.e. $\widehat{S}_{i,b}(t)$, could be negative. In this case $\widehat{S}_{i,b}$ follows approximately a geometric Brownian motion with negative initial condition, which is just minus a geometric Brownian motion with the opposite (positive) initial condition and the same volatility. The call option becomes then a put on the opposite geometric Brownian motion and has to be valued as such.

We may expect these formulas to work in all cases where the swaps in the portfolio all have the same direction, i.e. when all χ are equal to each other. In this case the underlying $\widehat{S}_{i,b}$ has always the same sign in all scenarios, and the approximation by a geometric Brownian motion is in principle reasonable.

In the other cases with mixed χ's (i.e. a portfolio with IRS both long and short), the underlying $\widehat{S}_{i,b}$ can be both positive and negative in different scenarios and at different times (even if it is still a martingale). In this case we approximate it with a geometric Brownian motion maintaining a constant sign equal to the sign of the initial condition and with the usual approximated volatility. We will see that results are not as bad as one can expect, provided some tricks are used. In particular, using put-call parity one has to set herself into the correct tail of the lognormal approximated density.

Indeed, consider for example as initial time $t = 0$ a case where $\widehat{S}_{i,b}(0)$ is positive but where the netting coefficients generate some negative future scenarios of $\widehat{S}_{i,b}(T_i)$. This way, the density of $\widehat{S}_{i,b}(T_i)$ will have both a positive and a negative tail. If we fit a lognormal distribution associated with a geometric Brownian motion with positive initial condition $\widehat{S}_{i,b}(0)$, when we price a call option on $\widehat{S}_{i,b}(T_i)$ both the true density and the approximated lognormal have the (right) tail, whereas if we price a put option we have the true underlying $\widehat{S}_{i,b}(T_i)$ with a left tail and the lognormal approximated process with no left tail. This means that the call will be priced in presence of tails both in the true underlying and in the approximated process, whereas the approximated process in the put case is missing the tail. Hence, from this point of view, it is best to price a call rather than a put. However if we do have to price a put, we can still price a call and apply the parity to get the put. This will result in a better approximation than integrating directly the put payoff against an approximated density that is missing the tail. We applied for example this method by computing the put prices in Section 8, Case A with the call price and the parity. The relative error we had obtained when integrating directly the put payoff in the money at $2y - 10y$ is -3.929% whereas applying the parity we obtained -2.156%.

Even so, at times precision will not be sufficient. We resort then to a method that takes into account also an approximated estimate of the third moment of the underlying $\widehat{S}_{i,b}$.

6.2 The three moments matching technique

As explained above a lognormal approximation may not be the right choice in the case of mixed (i.e. positive and negative) netting coefficients. In particular, linear combinations of lognormal variables with unit-weights (positive or negative) is no longer a lognormal.
In this case we have used the three moments matching technique (as in Brigo et al. (2003)) by shifting of a parameter X an auxiliary martingale lognormal process Y with a flag $\phi \in \{-1, +1\}$ (to consider the correct side of the distribution), leading to a dynamics of the form:

$$A_{T_i} = X + \phi\, Y(T_i) = X + \phi\, Y(t) \exp\left(\int_t^{T_i} \eta(s) dW_s - 1/2 \int_t^{T_i} \eta(s)^2 ds \right) \tag{6.15}$$

with W a Brownian motion under the $\widehat{C}_{i,b}$-measure and where η is the volatility of the process Y.
In particular we have:

$$\widehat{\mathbb{E}}_t^{i,b}[A_{T_i}] = X + \phi\, Y(t); \tag{6.16}$$

$$\widehat{\mathbb{E}}_t^{i,b}[(A_{T_i})^2] = X^2 + Y(t)^2 \exp\left(\int_t^{T_i} \eta(s)^2 ds \right) + 2\, \phi\, XY(t); \tag{6.17}$$

$$\widehat{\mathbb{E}}_t^{i,b}[(A_{T_i})^3] = X^3 + \phi\, Y(t)^3 \exp\left(3 \int_t^{T_i} \eta(s)^2 ds \right) + 3\, \phi\, X^2 Y(t) + 3XY(t)^2 \exp\left(\int_t^{T_i} \eta(s)^2 ds \right). \tag{6.18}$$

These non-central moments have to be matched against the first three moments of $\widehat{S}_{i,b}(T_i)$:

$$\widehat{\mathbb{E}}_t^{i,b}[(\widehat{S}_{i,b}(T_i))^m] = \tag{6.19}$$
$$= \sum_{j_1,\ldots,j_m=i+1}^{b} \widehat{w}_{j_1}(t)\ldots\widehat{w}_{j_m}(t)\chi_{j_1}\ldots\chi_{j_m} F_{j_1}(t)\ldots F_{j_m}(t) \exp\left\{ \sum_{k=i+1}^{m+i-1} \sum_{h=k+1}^{m+i} \rho_{j_k,j_h} \int_t^{T_i} \sigma_{j_k}(s)\sigma_{j_h}(s) ds \right\}$$

for $m = 1, 2, 3$.
Assuming η constant and taking $t = 0$ for simplicity, and solving analytically the system

$$\widehat{\mathbb{E}}_t^{i,b}[\widehat{S}_{i,b}(T_i)] = \widehat{\mathbb{E}}_t^{i,b}[A_{T_i}] \tag{6.20}$$

$$\widehat{\mathbb{E}}_t^{i,b}[(\widehat{S}_{i,b}(T_i))^2] = \widehat{\mathbb{E}}_t^{i,b}[(A_{T_i})^2] \tag{6.21}$$

$$\widehat{\mathbb{E}}_t^{i,b}[(\widehat{S}_{i,b}(T_i))^3] = \widehat{\mathbb{E}}_t^{i,b}[(A_{T_i})^3] \tag{6.22}$$

for $X, Y(0), \eta$, we can exploit the auxiliary process to approximate the price (6.13) by

$$\widehat{C}_{i,b}(0)\text{Black}(Y(0), \eta^2(T_i), (\widehat{K} - X)) \tag{6.23}$$

where the triplet $(Y(0), \eta^2(T_i), X)$ is the solution of the following system of equations:

$$(\exp(\eta^2(T_i)) - 1)^{1/2} = \frac{(-4\beta + 4\sqrt{4+\beta^2})^{1/3}}{2} - \frac{2}{(-4\beta + 4\sqrt{4+\beta^2})^{1/3}} \quad (6.24)$$

$$Y(0) = \sqrt{\frac{m_2 - m_1^2}{\exp(\eta^2(T_i)) - 1}} \quad (6.25)$$

$$X = m_1 + \phi Y(0) \quad (6.26)$$

for $\beta = \phi \frac{m_1(3m_2 - 2m_1^2) - m_3}{(m_2 - m_1^2)^{3/2}}$, and with (m_1, m_2, m_3) being the moments achieved by formula

$$m_n(T_i) = \sum_{j_1,\dots,j_n=i+1}^{b} \widehat{w}_{j_1}(0)\dots\widehat{w}_{j_n}(0)\chi_{j_1}\dots\chi_{j_n} F_{j_1}(0)\dots F_{j_n}(0) \exp\left\{ \sum_{k=i+1}^{n+i-1} \sum_{h=k+1}^{n+i} \rho_{j_k,j_h} \int_0^{T_i} \sigma_{j_k}(s)\sigma_{j_h}(s)ds \right\}$$

for $n = 1, 2, 3$.

This holds provided that $Y(0)$ and $(\widehat{K} - X)$ have the same sign. Otherwise, depending on the sign of the pair $(Y(0), \widehat{K} - X)$ we will have a forward on $Y(0)$ or a claim with zero present value (as illustrated in the previous discussion following Equation (6.13)).
The role of ϕ is to switch the distribution on the correct side of the mass-points which, once again, depend on sign of netting coefficients. Therefore, the ϕ is the switch-factor and the X is the shift-factor of our auxiliary process.

7 Numerical Tests: all swaps in the same direction

Here we report the results we have achieved by testing our approximation versus Monte Carlo simulation (MC). We set $t = T_a = 0$, $T_b = 10$ and $\beta_i = 0.25$ for each $i \in (a, b] = (0, 40]$. Then, for a fixed T_i, we have compared the expectation $\mathbb{E}_t[D(t, T_i)(\text{NPV}(T_i))^+]$ computed via MC and via a Black-like approximation for set of tests with different volatilities, instantaneous correlations, forward rates curve and for several schemes of netting coefficients α_i.

Our examples are built in such a way that $\chi_i = \psi_i$ for all $i \in (a, b]$.

In the following tables, **B** denotes the Black-like approximation formula (**3MM** the Black three moment matching approximation), MC the Monte Carlo simulation, CI the confidence interval 1.96*(MC Standard Error), B-MC (3MM-MC) the difference between **B** (**3MM**) and MC, %BM the relative difference $(\mathbf{B}/MC - 1) * 100$ $((\mathbf{3MM}/MC - 1) * 100)$.
Note that once the forward rate curve is changed (steepened upwards or parallel shifted by +200bp) then the swap rates and hence the $\widetilde{K}_i$'s have to change as well.
The check point T_i is fixed along the life of our portfolio.
Finally, for each test, in the first column we used the following notations:

σ,ρ,F : to indicate a test under initial market inputs;

2σ,ρ,F : to indicate a test with double volatilities with respect to the initial market inputs;

σ,ρ,$\overrightarrow{F}$: to indicate a test with the initial forward curve steepened upwards w.r.t. the initial market inputs;

σ,ρ,$\widetilde{F}$: to indicate a test with the initial forward curve shifted by +200bp w.r.t. the initial market inputs;

$\sigma,\rho \approx 1,F$: to indicate a test with instantaneous correlations close to 1.

In Brigo and Masetti (2005) we present all possible combinations of tests. Here we give some of the most relevant cases, including the best and worst performing examples.

7.1 Case A: IRS with common last payment on T_b

In this case we proceed with the following schemes of netting coefficients and strikes:

I : $\alpha_i = (T_i - T_a)$ for each $i \in (a, b]$ (where $T_\cdot$ is expressed in terms of year-fraction whereas all the $i's$ are integers), i.e. we are considering IRS's with increasing start date and with common maturities T_b.
Graphically it may be represented like that: the vertical number of stars "*" is the multiple in front of the LIBOR rate at each reset. So for example in this case at the first reset we only have one flow, at the second reset two flows and so on.

```
POSITIVE FLOW          |                     *
                       |                 * __ *
                       |             * __ * __ *
                       |         * __ * __ * __ *
                       |     * __ * __ * __ * __ *
                       | * __ * __ * __ * __ * __ *
-----------------------+---------------------------
INCREASING TENOR  T_i ⟶|
```

II : $\widetilde{K}_i = \beta_i \sum_{j=a}^{i-1} S_{j,b}(t)$ for each $i \in (a, b]$;

III : $S_{i,b}(t) = \frac{\sum_{j=i+1}^{b} \beta_j P(t,T_j) F_j(t)}{\sum_{j=i+1}^{b} \beta_j P(t,T_j)}$

TestA1	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.564613	0.002746	0.56672	0.002107	0.373176
$[\sigma,\rho,F]$	5y10y	0.680602	0.003811	0.68034	-0.00026	-0.0385
$[\sigma,\rho,F]$	8y10y	0.393573	0.00278	0.39438	0.000807	0.205045

TestA1: standard market inputs.

TestA2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	1.07034	0.001755	1.0799	0.00956	0.893174
$[2\sigma,\rho,F]$	5y10y	1.2291	0.002506	1.2377	0.0086	0.699699
$[2\sigma,\rho,F]$	8y10y	0.713387	0.001975	0.71503	0.001643	0.23031

TestA2: doubled volatilities.

7.2 Case B: IRS with common first resetting date T_a

In this case we proceed with the following schemes of netting coefficients and strikes:

I : $\alpha_i = (T_b + \beta_i - T_i)$ for each $i \in (a, b]$, i.e. for a portfolio of IRS's with decreasing tenor and same start date; that is:

POSITIVE FLOW	$*$
	$* \_\_ *$
	$* \_\_ * \_\_ *$
	$* \_\_ * \_\_ * \_\_ *$
	$* \_\_ * \_\_ * \_\_ * \_\_ *$
	$* \_\_ * \_\_ * \_\_ * \_\_ * \_\_ *$
INCREASING TENOR $T_i \longrightarrow$	

II : $\widetilde{K}_i = \beta_i \sum_{j=i}^{b} S_{a,j}(t)$ for each $i \in (a, b]$;

III : $S_{a,i}(t) = \frac{\sum_{j=a+1}^{i} \beta_j P(t,T_j) F_j(t)}{\sum_{j=a+1}^{i} \beta_j P(t,T_j)}$

TestB1	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.673734	0.002688	0.67721	0.003476	0.515931
$[\sigma,\rho,F]$	5y10y	0.386824	0.001781	0.38792	0.001096	0.283333
$[\sigma,\rho,F]$	8y10y	0.069669	0.000408	0.069615	-5.4E-05	-0.07737

TestB1: standard market inputs.

TestB2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	1.04435	0.001664	1.0532	0.00885	0.847417
$[2\sigma,\rho,F]$	5y10y	0.577499	0.001138	0.58102	0.003521	0.609698
$[2\sigma,\rho,F]$	8y10y	0.106133	0.000277	0.10643	0.000297	0.279838

TestB2: doubled volatilities.

8 Numerical Tests: Swaps in different directions

Instead of considering only positive netting coefficients, here we allow our portfolio to be long or short along its tenor.
We have two symmetric cases (A, B) and one asymmetric case (C) where we have considered a less conservative portfolio strategy.
Here we have included at, in and out of the money tests as well, by setting:

ATM at the money test: strike at $\widetilde{K}_i$, $i \in (a, b]$;

ITM in the money test: strike at $0.75\widetilde{K}_i$, $i \in (a, b]$;

OTM out of the money test: strike at $1.25\widetilde{K}_i$, $i \in (a, b]$.

As pointed out before, given the current structure of the netting coefficients, we have to test the MC simulation both versus the Black approximation and versus the Black three Moment Matching approximation (derived in Section 6.2).

8.1 Case A: ATM, ITM, OTM

In this case we proceed with the following schemes of netting coefficients and strikes:

I : $\alpha_i = (T_{b/2} + \beta_i - T_i)\mathbf{1}_{\{T_i \le T_{b/2}\}} - (T_i - T_{b/2} - T_a)\mathbf{1}_{\{T_i > T_{b/2}\}}$ for each $i \in (a, b]$; that is:

POSITIVE FLOW	*	
	* — *	
	* — * — *	
	* — * — * — *	
	* — * — * — * — *	
	* — * — * — * — * — *	
NEGATIVE FLOW		* — * — * — * — * — *
		* — * — * — * — *
		* — * — * — *
		* — * — *
		* — *
		*
INCREASING TENOR $T_i \longrightarrow$		

II : $\chi_i \widetilde{K}_i = \beta_i \sum_{j=i}^{b/2} S_{a,j}(t)\mathbf{1}_{\{T_i \le T_{b/2}\}} - \beta_i \sum_{j=b/2+1}^{i} S_{j,b}(t)\mathbf{1}_{\{T_i > T_{b/2}\}}$ for each $i \in (a, b]$;

III : $S_{i,b}(t) = \frac{\sum_{j=i+1}^{b} \beta_j P(t,T_j) F_j(t)}{\sum_{j=i+1}^{b} \beta_j P(t,T_j)}$;

IV : $S_{a,i}(t) = \frac{\sum_{j=a+1}^{i} \beta_j P(t,T_j) F_j(t)}{\sum_{j=a+1}^{i} \beta_j P(t,T_j)}$;

V : $\chi_i = \mathbf{1}_{\{T_i \le T_{b/2}\}} - \mathbf{1}_{\{T_i > T_{b/2}\}}$.

Case A has been obtained by exploiting the put-call parity[1].

ATM:

TestA1	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
[σ,ρ,F]	2y10y	0.1501014	0.000393	0.15149	0.001389	0.925108
[σ,ρ,F]	5y10y	0.189583	0.001151	0.18967	8.7E-05	0.04589
[σ,ρ,F]	8y10y	0.1481094	0.001186	0.14812	1.06E-05	0.007157
TestA1	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
[σ,ρ,F]	2y10y	0.1501014	0.000393	0.15122	0.001119	0.74523
[σ,ρ,F]	5y10y	0.189583	0.001151	0.1897	0.000117	0.061714
[σ,ρ,F]	8y10y	0.1481094	0.001186	0.14812	1.06E-05	0.007157

TestA1 ATM: standard market inputs.

[1] By the symmetry between Case A and Case B we used MC Case B and the forward values on $\widehat{S}_{i,b}$ to get MC Case A.

TestA2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.247454	0.000299	0.24694	-0.00051	-0.20772
$[2\sigma,\rho,F]$	5y10y	0.37552	0.000773	0.3765	0.00098	0.260971
$[2\sigma,\rho,F]$	8y10y	0.2954154	0.000868	0.2961	0.000685	0.231741
TestA2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.247454	0.000299	0.25069	0.003236	1.307718
$[2\sigma,\rho,F]$	5y10y	0.37552	0.000773	0.37663	0.00111	0.29559
$[2\sigma,\rho,F]$	8y10y	0.2954154	0.000868	0.29606	0.000645	0.218201

TestA2 ATM: doubled volatilities.

ITM:

TestA1	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.017351	0.000676	0.016977	-0.00037	-2.1555
$[\sigma,\rho,F]$	5y10y	0.024322	0.00159	0.024541	0.000219	0.900419
$[\sigma,\rho,F]$	8y10y	0.032865	0.001535	0.032859	-6E-06	-0.01826
TestA1	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.017351	0.000676	0.018907	0.001556	8.967783
$[\sigma,\rho,F]$	5y10y	0.024322	0.00159	0.02449	0.000168	0.690733
$[\sigma,\rho,F]$	8y10y	0.032865	0.001535	0.03281	-5.5E-05	-0.16735

TestA1 ITM: standard market inputs.

TestA2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.089582	0.000388	0.08282	-0.00676	-7.54839
$[2\sigma,\rho,F]$	5y10y	0.139803	0.000915	0.1405	0.000697	0.498559
$[2\sigma,\rho,F]$	8y10y	0.137311	0.000975	0.13767	0.000359	0.26145
TestA2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.089582	0.000388	0.092842	0.00326	3.639124
$[2\sigma,\rho,F]$	5y10y	0.139803	0.000915	0.14033	0.000527	0.376959
$[2\sigma,\rho,F]$	8y10y	0.137311	0.000975	0.13745	0.000139	0.10123

TestA2 ITM: doubled volatilities.

TestA4	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,\widetilde{F}]$	5y10y	0.029094	0.001991	0.02869	-0.0004	-1.3886
TestA4	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,\widetilde{F}]$	5y10y	0.029094	0.001991	0.028625	-0.00047	-1.61202

TestA4 ITM: forward rates curve shifted by +200bp.

TestA5	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho \approx 1,F]$	5y10y	0.030852	0.000533	0.030943	9.1E-05	0.294957
TestA5	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho \approx 1,F]$	5y10y	0.030852	0.000533	0.030794	-5.8E-05	-0.18799

TestA5 ITM: perfect correlations.

OTM:

TestA1	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.413263	0.000144	0.41434	0.001077	0.260674
$[\sigma,\rho,F]$	5y10y	0.536445	0.000626	0.53685	0.000405	0.075497
$[\sigma,\rho,F]$	8y10y	0.348977	0.000805	0.34906	8.32E-05	0.023841
TestA1	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.413263	0.000144	0.41321	-5.3E-05	-0.01276
$[\sigma,\rho,F]$	5y10y	0.536445	0.000626	0.53694	0.000495	0.092274
$[\sigma,\rho,F]$	8y10y	0.348977	0.000805	0.34912	0.000143	0.041034

TestA1 OTM: standard market inputs.

TestA3	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,\overrightarrow{F}]$	5y10y	0.650079	0.000782	0.65003	-4.9E-05	-0.0076
TestA3	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,\overrightarrow{F}]$	5y10y	0.650079	0.000782	0.65014	6.06E-05	0.009322

TestA3 OTM: forward rates curve tilted upwards with $\overrightarrow{F}_{a+1}(0) = F_{a+1}(0)$.

8.2 Case B: ATM, ITM, OTM

In this case we proceed with the following schemes of netting coefficients and strikes:

I : $\alpha_i = -(T_{b/2} + \beta_i - T_i)\mathbf{1}_{\{T_i \le T_{b/2}\}} + (T_i - T_{b/2} - T_a)\mathbf{1}_{\{T_i > T_{b/2}\}}$ for each $i \in (a, b]$; that is:

POSITIVE FLOW	*
	* — *
	* — * — *
	* — * — * — *
	* — * — * — * — *
	* — * — * — * — * — *
NEGATIVE FLOW	* — * — * — * — * — *
	* — * — * — * — *
	* — * — * — *
	* — * — *
	* — *
	*
INCREASING TENOR $T_i \longrightarrow$	

II : $\chi_i \widetilde{K}_i = -\beta_i \sum_{j=i}^{b/2} S_{a,j}(t)\mathbf{1}_{\{T_i \le T_{b/2}\}} + \beta_i \sum_{j=b/2+1}^{i} S_{j,b}(t)\mathbf{1}_{\{T_i > T_{b/2}\}}$ for each $i \in (a, b]$;

III : $S_{i,b}(t) = \frac{\sum_{j=i+1}^{b} \beta_j P(t,T_j) F_j(t)}{\sum_{j=i+1}^{b} \beta_j P(t,T_j)}$;

IV : $S_{a,i}(t) = \frac{\sum_{j=a+1}^{i} \beta_j P(t,T_j) F_j(t)}{\sum_{j=a+1}^{i} \beta_j P(t,T_j)}$;

V : $\chi_i = -\mathbf{1}_{\{T_i \le T_{b/2}\}} + \mathbf{1}_{\{T_i > T_{b/2}\}}$.

Case B is symmetric compared to Case A.

ATM:

TestB1	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.059502	0.000393	0.06089	0.001388	2.332007
$[\sigma,\rho,F]$	5y10y	0.189583	0.001151	0.18967	8.7E-05	0.04589
$[\sigma,\rho,F]$	8y10y	0.155868	0.001186	0.15588	1.2E-05	0.007699
TestB1	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.059502	0.000393	0.060626	0.001124	1.888327
$[\sigma,\rho,F]$	5y10y	0.189583	0.001151	0.1897	0.000117	0.061714
$[\sigma,\rho,F]$	8y10y	0.155868	0.001186	0.15588	1.2E-05	0.007699

TestB1 ATM: standard market inputs.

TestB2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.156855	0.000299	0.15635	-0.00051	-0.32195
$[2\sigma,\rho,F]$	5y10y	0.37552	0.000773	0.3765	0.00098	0.260971
$[2\sigma,\rho,F]$	8y10y	0.303174	0.000868	0.30386	0.000686	0.226273
TestB2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.156855	0.000299	0.16009	0.003235	2.062414
$[2\sigma,\rho,F]$	5y10y	0.37552	0.000773	0.37663	0.00111	0.29559
$[2\sigma,\rho,F]$	8y10y	0.303174	0.000868	0.30381	0.000636	0.209781

TestB2 ATM: doubled volatilities.

ITM:

TestB1	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.241171	0.000676	0.2408	-0.00037	-0.15383
$[\sigma,\rho,F]$	5y10y	0.508562	0.00159	0.50878	0.000218	0.042866
$[\sigma,\rho,F]$	8y10y	0.332985	0.001535	0.33298	-5E-06	-0.0015
TestB1	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.241171	0.000676	0.24273	0.001559	0.646429
$[\sigma,\rho,F]$	5y10y	0.508562	0.00159	0.50873	0.000168	0.033034
$[\sigma,\rho,F]$	8y10y	0.332985	0.001535	0.33293	-5.5E-05	-0.01652

TestB1 ITM: standard market inputs.

TestB2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.313402	0.000388	0.30665	-0.00675	-2.15442
$[2\sigma,\rho,F]$	5y10y	0.624043	0.000915	0.62474	0.000697	0.111691
$[2\sigma,\rho,F]$	8y10y	0.437431	0.000975	0.4378	0.000369	0.084356
TestB2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.313402	0.000388	0.31667	0.003268	1.04275
$[2\sigma,\rho,F]$	5y10y	0.624043	0.000915	0.62457	0.000527	0.084449
$[2\sigma,\rho,F]$	8y10y	0.437431	0.000975	0.43757	0.000139	0.031776

TestB2 ITM: doubled volatilities.

OTM:

TestB1	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.008243	0.000144	0.009317	0.001074	13.02687
$[\sigma,\rho,F]$	5y10y	0.052205	0.000626	0.05261	0.000405	0.775788
$[\sigma,\rho,F]$	8y10y	0.064367	0.000805	0.064456	8.92E-05	0.138581
TestB1	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.008243	0.000144	0.008185	-5.8E-05	-0.70644
$[\sigma,\rho,F]$	5y10y	0.052205	0.000626	0.052694	0.000489	0.936692
$[\sigma,\rho,F]$	8y10y	0.064367	0.000805	0.064509	0.000142	0.220921

TestB1 OTM: standard market inputs.

TestB2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.073993	0.000212	0.076245	0.002252	3.043392
$[2\sigma,\rho,F]$	5y10y	0.221966	0.00062	0.22318	0.001214	0.546931
$[2\sigma,\rho,F]$	8y10y	0.212177	0.000761	0.21272	0.000543	0.255918
TestB2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.073993	0.000212	0.074158	0.000165	0.222859
$[2\sigma,\rho,F]$	5y10y	0.221966	0.00062	0.22354	0.001574	0.709118
$[2\sigma,\rho,F]$	8y10y	0.212177	0.000761	0.21285	0.000673	0.317188

TestB2 OTM: doubled volatilities.

8.3 Case C: ATM, ITM, OTM

In this case we proceed with the following schemes of netting coefficients and strikes:

I : $\alpha_i = (-1)^{i+1}(T_b + \beta_i - T_i)$ for each $i \in (a, b]$; that is:

POSITIVE FLOW	*					
	*					
	*		*			
	*		*			
	*		*		*	
	*		*		*	
NEGATIVE FLOW		*		*		*
		*		*		
		*		*		
		*				
		*				
INCREASING TENOR $T_i \longrightarrow$						

II : $\chi_i \widetilde{K}_i = (-1)^{i+1} \beta_i \sum_{j=i}^{b} S_{a,j}(t)$ for each $i \in (a, b]$;

III : $S_{a,i}(t) = \frac{\sum_{j=a+1}^{i} \beta_j P(t,T_j) F_j(t)}{\sum_{j=a+1}^{i} \beta_j P(t,T_j)}$;

IV : $\chi_i = (-1)^{i+1}$.

Case C is completely asymmetric hence put-call parity is no longer applied.
Actually Case C is a special case of Case B with common first resetting date (described in section 7.2) but with long and short position which switch along the tenor of our portfolio.

ATM:

TestC1	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.017589	0.000101	0.017398	-0.00019	-1.08422
$[\sigma,\rho,F]$	5y10y	0.016347	0.000101	0.016079	-0.00027	-1.64005
$[\sigma,\rho,F]$	8y10y	0.007337	4.94E-05	0.007325	-1.2E-05	-0.16573
TestC1	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.017589	0.000101	0.017545	-4.4E-05	-0.24845
$[\sigma,\rho,F]$	5y10y	0.016347	0.000101	0.016255	-9.2E-05	-0.5634
$[\sigma,\rho,F]$	8y10y	0.007337	4.94E-05	0.007348	1.05E-05	0.143648

TestC1 ATM: standard market inputs.

TestC2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.036143	7.59E-05	0.035204	-0.00094	-2.59855
$[2\sigma,\rho,F]$	5y10y	0.029971	7.70E-05	0.028942	-0.00103	-3.433
$[2\sigma,\rho,F]$	8y10y	0.012224	3.71E-05	0.012083	-0.00014	-1.15509
TestC2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.036143	7.59E-05	0.036503	0.00036	0.995485
$[2\sigma,\rho,F]$	5y10y	0.029971	7.70E-05	0.030137	0.000166	0.554204
$[2\sigma,\rho,F]$	8y10y	0.012224	3.71E-05	0.012231	6.8E-06	0.055627

TestC2 ATM: doubled volatilities.

TestC3	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,\overrightarrow{F}]$	5y10y	0.021802	0.000135	0.021471	-0.00033	-1.51595
TestC3	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,\overrightarrow{F}]$	5y10y	0.021802	0.000135	0.021717	-8.5E-05	-0.38759

TestC3 ATM: forward rates curve tilted upwards with $\overrightarrow{F}_{a+1}(0) = F_{a+1}(0)$.

TestC4	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,\widetilde{F}]$	5y10y	0.020228	0.00014	0.020002	-0.00023	-1.1158
TestC4	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,\widetilde{F}]$	5y10y	0.020228	0.00014	0.020182	-4.6E-05	-0.22593

TestC4 ATM: forward rates curve shifted by +200bp.

ITM:

TestC1	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.041527	0.000137	0.041063	-0.00046	-1.11758
$[\sigma,\rho,F]$	5y10y	0.030593	0.000124	0.030207	-0.00039	-1.26205
$[\sigma,\rho,F]$	8y10y	0.012757	5.80E-05	0.012744	-1.3E-05	-0.09956
TestC1	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.041527	0.000137	0.041482	-4.5E-05	-0.1086
$[\sigma,\rho,F]$	5y10y	0.030593	0.000124	0.030454	-0.00014	-0.45468
$[\sigma,\rho,F]$	8y10y	0.012757	5.80E-05	0.012769	1.23E-05	0.09642

TestC1 ITM: standard market inputs.

TestC2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.055429	8.76E-05	0.053473	-0.00196	-3.52849
$[2\sigma,\rho,F]$	5y10y	0.0407	8.46E-05	0.039227	-0.00147	-3.61964
$[2\sigma,\rho,F]$	8y10y	0.016122	4.02E-05	0.015923	-0.0002	-1.23679
TestC2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.055429	8.76E-05	0.055487	5.82E-05	0.105
$[2\sigma,\rho,F]$	5y10y	0.0407	8.46E-05	0.040647	-5.3E-05	-0.13071
$[2\sigma,\rho,F]$	8y10y	0.016122	4.02E-05	0.01609	-3.2E-05	-0.20096

TestC2 ITM: doubled volatilities.

TestC3	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,\overrightarrow{F}]$	5y10y	0.03946	0.000163	0.039052	-0.00041	-1.0327
TestC3	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,\overrightarrow{F}]$	5y10y	0.03946	0.000163	0.039364	-9.5E-05	-0.24202

TestC3 ITM: forward rates curve tilted upwards with $\overrightarrow{F}_{a+1}(0) = F_{a+1}(0)$.

OTM:

TestC1	T_i	MC (400K paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.005991	6.12E-05	0.006199	0.000208	3.474119
$[\sigma,\rho,F]$	5y10y	0.007945	7.48E-05	0.007883	-6.2E-05	-0.77584
$[\sigma,\rho,F]$	8y10y	0.003923	3.86E-05	0.003916	-6.7E-06	-0.17182
TestC1	T_i	MC (400K paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho,F]$	2y10y	0.005991	6.12E-05	0.005998	7.14E-06	0.119176
$[\sigma,\rho,F]$	5y10y	0.007945	7.48E-05	0.00789	-5.5E-05	-0.69654
$[\sigma,\rho,F]$	8y10y	0.003923	3.86E-05	0.003922	-2.4E-07	-0.00612

TestC1 OTM: standard market inputs.

TestC2	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.023315	6.42E-05	0.023247	-6.8E-05	-0.29337
$[2\sigma,\rho,F]$	5y10y	0.022213	6.98E-05	0.021605	-0.00061	-2.73626
$[2\sigma,\rho,F]$	8y10y	0.009391	3.41E-05	0.009268	-0.00012	-1.30774
TestC2	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[2\sigma,\rho,F]$	2y10y	0.023315	6.42E-05	0.023708	0.000393	1.683866
$[2\sigma,\rho,F]$	5y10y	0.022213	6.98E-05	0.022465	0.000252	1.135381
$[2\sigma,\rho,F]$	8y10y	0.009391	3.41E-05	0.009382	-8.8E-06	-0.09381

TestC2 OTM: doubled volatilities.

TestC5	T_i	MC (4M paths)	CI	**B**	B-MC	%BM
$[\sigma,\rho \approx 1,F]$	5y10y	0.005087	1.65E-05	0.004942	-0.00015	-2.86004
TestC5	T_i	MC (4M paths)	CI	**3MM**	3MM-MC	%BM
$[\sigma,\rho \approx 1,F]$	5y10y	0.005087	1.65E-05	0.004954	-0.00013	-2.61825

TestC5 OTM: perfect correlations.

8.4 Conclusions on Part II

We introduced counterparty risk formulas for IRS, also under netting agreements. In the latter case we derived two approximated formulas and tested both of them against Monte Carlo simulation, finding a good agreement under most market configurations.

More in detail, as expected, the Black like approximation works well in the case of netting coefficients going into a single direction. When we consider a portfolio with positive and negative coefficients results are not as good, particularly for “in the money” and “out of the money” strikes. However, in general the more refined formula (Black three Moment Matching approximation, shifted lognormal distribution) outperforms the standard Black approximation (lognormal distribution). This result does not hold for the Case A/ITM, described in Section 8.1, where the Black three Moment Matching formula does not outperform the simpler Black approximation. We note however that both results are still within the Monte Carlo window given by the standard error.

There are several cases where the moment matching brings in a considerable improvement with respect to the basic Black formula. For example Case B Section 8.2, OTM TestB1, and case C (characterized by asymmetric coefficients) both for ATM, ITM and OTM Tests.

The possibility to include the netting agreement lowers considerably the price component due to counterparty risk. In fact, in absence of the correct implementation of netting coefficients formulas we would obtain just the sum of counterparty risk pricing in each single IRS, meaning that we are counting multiple times the default impact of flows that are more than in a single IRS. In a way it is like pricing a payoff given by a sum of positive parts by doing the pricing of each positive part and then adding up. Since

$$(\Pi_{\mathrm{IRS}_1} + \Pi_{\mathrm{IRS}_2} + ... + \Pi_{\mathrm{IRS}_n})^+ \leq \Pi^+_{\mathrm{IRS}_1} + \Pi^+_{\mathrm{IRS}_2} + ... + \Pi^+_{\mathrm{IRS}_n}$$

(the left hand side corresponding to a netted portfolio of residual NPV’s) we see that the price under netting agreements is always smaller than the price with no netting agreement. Since this option component (times L_{GD}) is subtracted from the default free value to determine the counterparty risk price, we subtract more in absence of netting agreements; netting agreements produce a smaller expected loss in general, therefore the total value of the claim is larger. The interested reader can use our approximated formula (6.9, 6.13) or the more refined (6.9, 6.23) to check cases with different given curves of default probabilities to assess the typical impact of netting agreements in different default probability configurations, all the needed tools have been given in this part.

Part III

Counterparty Risk in Equity Payoffs

The second example we present in this Part III of the paper deals with counterparty risk pricing in the equity market. In this part we develop a few families of tractable structural models with analytical default probabilities depending on some dynamics parameters and on a possibly random default barrier and volatility ideally associated with the underlying firm debt. We follow Brigo and Tarenghi (2004, 2005).

The first family of models is simply an extension to the time-varying volatility case of the classic Black Cox (1976) structural first passage model. The time varying volatility of the firm value is used as a calibration parameter to exactly reproduce CDS quotes for different maturities. With this approach there is no calibration error, and we finally have a family of structural models with a calibration capability comparable to that of the rival and more tractable family of reduced form models seen earlier in Part II. This first family of structural models uses analytic barrier option results under time varying dynamics coefficients, based on Lo et al (2003) and Rapisarda (2003). We term the resulting model Analytically Tractable 1st Passage Model (AT1P), and introduce it in Section 9. We show a calibration example with Vodafone in Section 9.1, while we refer to Brigo and Tarenghi (2004) for a more extensive example involving Parmalat CDS data at several dates, where we show that the calibration capability of the AT1P structural model is considerable. We will also point out some limits of AT1P, especially as far as realistic values of volatilities and default barrier are concerned. These limits prompt us to introduce in Section 10 a random Scenario-based volatility/ default-barrier version of AT1P (called SVBAT1P) where we keep the volatility constant in time in each scenario. This constancy will cause the model to have less calibration power but more realistic outputs. The volatility scenarios will induce a mixture distribution on the firm value, while maintaining analytical tractability. The mixture dynamics is known to allow for fat tails, so that the task of spreading the trajectories now in order to hit the default barrier will not be entirely loaded into a lognormal volatility, avoiding unrealistically large volatilities as in the AT1P model. A calibration example with this extended model is given in Section 10.1.

The AT1P and SVBAT1P models are suited to evaluating counterparty risk in equity payoffs, and more generally to evaluate hybrid credit/equity payoffs, as we explain also in Brigo and Tarenghi (2004, 2005), by preserving the Black-Scholes models tractability. This theme is important also because of the implications of the Basel II framework concerning counterparty risk.

As an illustrative case we consider an example of counterparty risk pricing for an equity return swap (ERS). This is an interesting choice since the value of this contract is all due to counterparty risk: as we shall see, without counterparty risk the fair spread for this contract is null. The example is introduced in Section 11 and we show the ERS valuation under the different families of models considered.

Finally, we summarize our results and hint at further research in the conclusions.

9 The deterministic barrier AT1P model

The fundamental hypothesis of the model we resume here is that the underlying firm value process V is a Geometric Brownian Motion (GBM), which is also the kind of process commonly used for equity stocks in the Black Scholes model.

Classical structural models (Merton (1974), Black Cox (1976)) postulate a GBM (Black and Scholes) lognormal dynamics for the value of the firm V. This lognormality assumption is considered to be acceptable. Crouhy et al (2000) report that "this assumption is quite robust and, according to KMVs own empirical studies, actual data conform quite well to this hypothesis.".

In these models the value of the firm V is the sum of the firm equity value S and of the firm debt value D. The firm equity value S, in particular, can be seen as a kind of (vanilla or barrier-like) option on the value of the firm V. This link is important also when in need of pricing hybrid equity/credit products, such as equity default swaps. Merton typically assumes a zero-coupon debt at a terminal maturity $\bar{T}$. Black Cox assume, besides a possible zero coupon debt, safety covenants forcing the firm to declare bankruptcy and pay back its debt with what is left as soon as the value of the firm itself goes below a "safety level" barrier. This is what introduces the need for barrier option technology in structural models for default.

In Brigo and Tarenghi (2004) the following proposition is proved:

Proposition 9.1. (Analytically-Tractable First Passage (AT1P) Model) *Assume the risk neutral dynamics for the value of the firm V is characterized by a risk free rate $r(t)$, a payout ratio $q(t)$ and an instantaneous volatility $\sigma(t)$, all deterministic, according to*

$$dV(t) = V(t)\,(r(t) - q(t))\,dt + V(t)\,\sigma(t)\,dW(t)$$

and assume a safety barrier $\widehat{H}(t)$ of the form

$$\widehat{H}(t) = H \exp\left(-\int_0^t \left(q(s) - r(s) + (1+2\beta)\frac{\sigma(s)^2}{2}\right) ds\right) \tag{9.1}$$

where β is a parameter that can be used to shape the safety barrier, H is a reference initial level for the curved barrier, and let τ be defined as the first time where V hits the safety covenants barrier $\widehat{H}$ from above, starting from $V_0 > H$,

$$\tau = \inf\{t \geq 0 : V(t) \leq \widehat{H}(t)\}.$$

Then the survival probability is given analytically by

$$\mathbb{Q}\{\tau > T\} = \left[\Phi\left(\frac{\log\frac{V_0}{H} + \beta\int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}}\right) - \left(\frac{H}{V_0}\right)^{2\beta} \Phi\left(\frac{\log\frac{H}{V_0} + \beta\int_0^T \sigma(s)^2 ds}{\sqrt{\int_0^T \sigma(s)^2 ds}}\right)\right]. \tag{9.2}$$

The proposition is written with no reference to a possible zero coupon debt. In presence of zero coupon debt for the maturity $\bar{T}$, the actual default time is τ if $\tau < \bar{T}$, and is $\bar{T}$ if $\tau \geq \bar{T}$ and the value of the firm at $\bar{T}$ is below the debt face value. Otherwise, if $\tau \geq \bar{T}$ and the value of the firm at $\bar{T}$ is above the debt face value, there is no default. The proposition holds for $T < \bar{T}$.

We now recall our earlier introduction of the CDS payoff and its risk neutral pricing formula, in Section 3.

Recall Formula (3.3) for our CDS price. In our structural model setup it suffices to substitute the survival probabilities formula $t \mapsto \mathbb{Q}(\tau > t)$ of the AT1P or SVBAT1P structural models to value CDS.

As before, the idea is to use quoted values of the fair R's with increasing maturities T_b (and initial resets all set to $T_a = 0$) to derive the default probabilities assessed by the market.

To calibrate the AT1P model to quoted market R's for different CDS, we insert the quoted R's in formula (3.3) with survival probabilities given by (9.2) and find the $t \mapsto \sigma(t)$ and H values that set said formula to zero. See Brigo and Tarenghi (2004) for several numerical examples and a case study on Parmalat CDS data. A final remark concerns the well known fact that in all our survival and default probability formulas V_0 and H always appear combined in a ratio, so that we need only to worry about V_0/H and not on V_0 and H separately. We thus often assume a unit V_0 and express H with respect to unity.

9.1 AT1P Calibration: Numerical examples

Here we plan to analyze how the structural model behaves in a calibration to real CDS data. Some important remarks on the data. We have set the payout ratio $q(t)$ identically equal to zero. We present the calibration performed with the AT1P structural model to CDS contracts having Vodafone as underlying with recovery rate $\text{R}_{\text{EC}} = 40\%$ ($\text{L}_{\text{GD}} = 0.6$). In Table 1 we report the maturities T_b of the contracts and the corresponding "mid" CDS rates $R^{\text{MID}}_{0,b}(0)$ (quarterly paid) on the date of March 10, 2004, in basis points ($1bp = 10^{-4}$). We take $T_a = 0$ in all cases.

	CDS maturity T_b	$R^{\text{BID}}_{0,b}(0)$ (bps)	$R^{\text{ASK}}_{0,b}(0)$	$R^{\text{MID}}_{0,b}(0)$
1y	20-Mar-05	19	24	21.5
3y	20-Mar-07	32	34	33
5y	20-Mar-09	42	44	43
7y	20-Mar-11	45	53	49
10y	20-Mar-14	56	66	61

Table 1: Vodafone CDS quotes on March 10, 2004.

In Table 2 we report the values (in basis points) of the CDS's computed inserting the bid and ask premium rate R quotes into the payoff and valuing the CDS with hazard rates stripped by mid quotes. This way we transfer the bid offer spread in the rates R on a bid offer spread on the CDS payoff present value. In Table 3 we present the results of the calibration performed with the structural model and, as a comparison, of the calibration performed with a hazard rate model (postulating a piecewise linear hazard rate). In this first example the parameters used for the structural model have been selected on qualitative considerations, and are $q = 0$, $\beta = 0.5$ and $H/V_0 = 0.4$ (this is a significant choice since this value is in line

CDS mat T_b	$\text{CDS}_{0,b}$ value bid (bps)	$\text{CDS}_{0,b}$ value ask (bps)
1y	2.56	-2.56
3y	2.93	-2.93
5y	4.67	-4.67
7y	24.94	-24.94
10y	41.14	-41.14

Table 2: CDS values computed with hazard rates stripped from mid R Vodafone quotes but with bid and ask rates R in the premium legs.

with the expected value of the random H, completely determined by market quotes, in the Scenario based model presented later on).

We report the values of the calibrated parameters in the two models (volatilities and hazard rate nodes) and the survival probabilities, that appear to be very close under the two different models.

T_i	$\sigma(T_{i-1} \div T_i)$	Surv. $\mathbb{Q}(\tau > T_i)$ AT1P	Hazard rate
0	32.625%	100.000%	0.357
1y	32.625%	99.625%	0.357
3y	17.311%	98.315%	0.952
5y	17.683%	96.353%	1.033
7y	17.763%	94.206%	1.189
10y	21.861%	89.650%	2.104

Table 3: Results of the calibrations.

Further comments on the realism of short term credit spreads and on the robustness of default probabilities with respect to CDS are in Brigo and Tarenghi (2004).

10 The scenario volatility/barrier SVBAT1P model

In some cases it can be interesting to keep the volatility of the process V as an exogenous input coming from the equity and debt worlds (for example it could be related to an historical or implied volatility). Or we might retain a time-varying volatility to be used only partly as a fitting parameter. Or, also, we might wish to remove time-varying volatility to avoid an all-fitting approach that is dangerous for robustness.

In all cases, having lost degrees of freedom in the volatility, we would need to introduce other fitting parameters into the model. One such possibility comes from introducing a random default barrier. This corresponds to the intuition that the balance sheet information is not certain, possibly because the company is hiding some information. In this sense, assuming a random default barrier can model this uncertainty. This means that we retain the same model as before, but now the default barrier level H is replaced by a random variable H assuming different scenarios with given risk neutral probabilities. At a second stage, we may introduce volatility scenarios as well. By imposing time-constant volatility scenarios, in each single scenario we lose flexibility with respect to AT1P, but regain flexibility thanks to the multiple scenarios on otherwise too simple time-constant volatilities. Even so, the scenario based model results in a less flexible structure than the old deterministic volatility and barrier AT1P model as far as CDS calibration is concerned.

In detail:

Definition 10.1. (Scenario Volatility and Barrier Analytically Tractable 1st Passage model, SVBAT1P) *Let the firm value process risk neutral dynamics be given by*

$$dV(t) = (r(t) - q(t))V(t)dt + \nu(t)V(t)dW(t),$$

same notation as earlier in the paper. This time, however, let the safety barrier parameter H in (9.1) and the firm value volatility function $t \mapsto \nu(t)$ assume time-constant scenarios $(H_1, \sigma^1), \ldots, (H_{N-1}, \sigma^{N-1}), (H_N, \sigma^N)$ with $\mathbb{Q}$ probability $p_1, \ldots, p_{N-1}, p_N$ respectively. The safety barrier will thus be random and equal to

$$\widehat{H}^i(t) := H_i \exp\left(- \int_0^t \left(q(s) - r(s) + (1 + 2\beta)\frac{(\sigma^i)^2}{2} \right) ds \right)$$

with probability p_i. The random variables H and ν are assumed to be independent of the driving Brownian motion W of the value of the firm V risk neutral dynamics.

This definition has very general consequences. Indeed, if we are to price a payoff Π based on V, by iterated expectation we have

$$\mathbb{E}[\Pi] = \mathbb{E}\{\mathbb{E}[\Pi | H, \nu]\} = \sum_{i=1}^{N} p_i \mathbb{E}[\Pi | H = H_i, \nu = \sigma^i]$$

Now, thanks to independence, the term $\mathbb{E}[\Pi | H = H_i, \nu = \sigma^i]$ is simply the price of the payoff Π under the model with deterministic barrier and volatility seen earlier in the paper, when the barrier parameter H is set to H_i and the volatility to σ^i, so that the safety barrier is $\widehat{H}^i$. This means that, in particular, for CDS payoffs we obtain

$$\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}) = \sum_{i=1}^{N} \text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}; H_i, \sigma^i) \cdot p_i \tag{10.1}$$

where $\text{CDS}_{a,b}(0, R, \text{L}_{\text{GD}}; H_i, \sigma^i)$ is the CDS price (3.3) computed according to survival probabilities (9.2) when the barrier H is set to H_i and the volatility to σ^i. Let us consider now a set of natural maturities for CDS quotes. This is to say that we assume $T_a = 0$ and T_b ranging a set of standard maturities, $T_b = 1y, 3y, 5y, 7y, 10y$.

Now assume we aim at calibrating the scenario based barrier/volatility model to a term structure of CDS data.

In Brigo and Tarenghi (2005) some calibration experiments based on linear algebra and aiming at an exact calibration are presented. Here, however, we move directly to optimization.

10.1 Scenario based Barrier and Volatility: Numerical Optimization

We consider all of the five quotes of Table 1 and we use the general model with two scenarios on the barrier/volatility parameters (H, ν) and one probability (the other one being determined by normalization to one), with a total of five parameters for five quotes. Here, by trial and error we decided to set $\beta = 0$.

$$[H_1^*, H_2^*; \sigma_*^1, \sigma_*^2; p_1^*] = \text{argmin}_{H,p,\nu} \sum_{k=1}^{5} \Big[p_1 \text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_1, \sigma^1) + (1-p_1)\text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}(0), \text{L}_{\text{GD}}; H_2, \sigma^2) \Big]^2$$

H_i	σ_i	p_i
0.3721	17.37%	93.87%
0.6353	23.34%	6.13%

Table 4: SVBAT1P model calibrated to Vodafone CDS. The expected value of the barrier is $\mathbb{E}[H] = 0.3882$.

We obtain a low optimization error, i.e. $147bps^2$, corresponding to the following calibration errors on single CDS present values:

CDS maturity T_k	$\text{CDS}_{0,k}(0, R_{0,k}^{\text{MID}}, H_1, H_2; \sigma^1, \sigma^2; p_1)$ (bps)
1y	1.38
3y	-3.89
5y	8.16
7y	-7.56
10y	2.41

Table 5: CDS values obtained using the parameters resulting from the calibration.

The single CDS calibration errors are low, being the CDS present values corresponding to market R close to zero, with the exception of the five years maturity, which can be adjusted by introducing weights in the target function. Compare also with Table 2 to compare the calibration error with the CDS-induced bid ask spreads.

As a second attempt we introduce weights that are inversely proportional to the bid-ask spread and repeat the calibration for the SVBAT1P model finding the results presented in Tables 6 and 7 (compare with Tables 4 and 5).

H_i	σ_i	p_i
0.3713	17.22%	92.63%
0.6239	22.17%	7.37%

Table 6: SVBAT1P model calibrated to Vodafone CDS using weights in the objective function which are inversely proportional to the bid ask-spread. The expected value of the barrier is $\mathbb{E}[H] = 0.3899$.

We see that now the 1y CDS value is outside the bid-ask spread, but conversely the 5y CDS value has decreased, assuming a size more in line with the spread. This is what we

CDS maturity T_k	$\text{CDS}_{0,k}(0, R^{\text{MID}}_{0,k}, H_1, H_2; \sigma^1, \sigma^2; p_1)$ (bps)
1y	5.85
3y	-3.76
5y	4.92
7y	-10.46
10y	1.47

Table 7: CDS values obtained using the parameters resulting from the weighted calibration.

aimed at, since the 5y CDS is probably the most liquid one and the model needs to reproduce it well.

More in general, when combining default barrier and volatility scenarios, scenarios on ν and H can be taken jointly (as we did so far) or separately, but one needs to keep the combinatorial explosion under control. The pricing formula remains easy, giving linear combination of formulas in each basic scenario. Taking time-varying parametric forms for the σ^i's can add flexibility and increase the calibrating power of the model, and will be addressed in future work.

A final important remark is in order. We see that the parameters resulting from the scenario versions calibration are more credible than those obtained in the AT1P framework. In particular we notice that in all cases we have an expected H which is comparable with the fixed $H = 0.4$ used in the deterministic case. What is more, even if the H are similar, the volatilities involved are smaller than in the AT1P case. As previously hinted at, this fact is essentially due to volatility scenarios inducing a mixture of lognormal distributions for the firm value, implying fatter tails, and allowing for the same default probabilities with smaller volatilities.

11 Counterparty risk in equity return swaps

This section summarizes the results on counterparty risk pricing in Equity Return Swaps under AT1P in Brigo and Tarenghi (2004) and under S(V)BAT1P in Brigo and Tarenghi (2005). This is an example of counterparty risk pricing with the calibrated structural model in the equity market. This method can be easily generalized to different equity payoffs.

Let us consider an equity return swap payoff. Assume we are a company "A" entering a contract with company "B", our counterparty. The reference underlying equity is company "C". The contract, in its prototypical form, is built as follows. Companies "A" and "B" agree on a certain amount K of stocks of a reference entity "C" (with price $S = S^C$) to be taken as nominal ($N = K\,S_0$). The contract starts in $T_a = 0$ and has final maturity $T_b = T$. At $t = 0$ there is no exchange of cash (alternatively, we can think that "B" delivers to "A" an amount K of "C" stock and receives a cash amount equal to KS_0). At intermediate times "A" pays to "B" the dividend flows of the stocks (if any) in exchange for a periodic rate (for example a semi-annual LIBOR rate L) plus a spread X. At final maturity $T = T_b$, "A" pays KS_T to "B" (or gives back the amount K of stocks) and receives a payment KS_0. This can be summarized as follows:

Initial Time 0: no flows, or
A $\longrightarrow KS_0^C$ cash $\longrightarrow$ B
A $\longleftarrow K$ equity of "C" $\longleftarrow$ B
....
Time T_i:
A $\longrightarrow$ equity dividends of "C" $\longrightarrow$ B
A $\longleftarrow$ Libor + Spread $\longleftarrow$ B
....
Final Time T_b:
A $\longrightarrow$ K equity of "C" $\longrightarrow$ B
A $\longleftarrow KS_0^C$ cash $\longleftarrow$ B

The price of this product can be derived using risk neutral valuation, and the (fair) spread is chosen in order to obtain a contract whose value at inception is zero. We ignore default of the underlying "C", thus assuming it has a much stronger credit quality than the counterparty "B", that remains our main interest. It can be proved that if we do not consider default risk for the counterparty "B" either, the fair spread is identically equal to zero. This renders the ERS an interesting contract since all its value is due to counterparty risk. Indeed, when taking into account counterparty default risk in the valuation the fair spread is no longer zero. In case an early default of the counterparty "B" occurs, the following happens. Let us call $\tau = \tau_B$ the default instant. Before τ everything is as before, but if $\tau \leq T$, the net present value (NPV) of the position at time τ is computed. If this NPV is negative for us, i.e. for "A", then its opposite is completely paid to "B" by us at time τ itself. On the contrary, if it is positive for "A", it is not received completely but only a recovery fraction R_{EC} of that NPV is received by us. It is clear that to us ("A") the counterparty risk is a problem when the NPV is large and positive, since in case "B" defaults we receive only a fraction of it.

The risk neutral expectation of the discounted payoff is given in the following proposition (see e.g. Brigo and Tarenghi (2004), $L(S,T)$ is the simply compounded rate at time S for maturity T):

Proposition 11.1. (Equity Return Swap price under Counterparty Risk). *The fair price of the Equity Return Swap defined above can be simplified as follows:*

$$ERS(0) = KS_0 X \sum_{i=1}^{b} \alpha_i P(0,T_i) - L_{GD}\mathbb{E}_0\Big\{\mathbf{1}_{\{\tau \leq T_b\}} D(0,\tau)(NPV(\tau))^+\Big\}.$$

where

$$\begin{aligned} NPV(\tau) &= \mathbb{E}_\tau\Big\{ -K\,NPV_{dividends}^{\tau \div T_b}(\tau) + KS_0 \sum_{i=\beta(\tau)}^{b} D(\tau,T_i)\alpha_i\left(L(T_{i-1},T_i) + X\right) \\ &\quad + \left(KS_0 - KS_{T_b}\right) D\left(\tau, T_b\right)\Big\}. \end{aligned} \tag{11.1}$$

and where we denote by $NPV_{dividends}^{s \div t}(u)$ the net present value of the dividend flows between s and t computed in u.

The first term in $ERS(0)$ is the equity swap price in a default-free world, whereas the second one is the optional price component due to counterparty risk, see the general Formula 4.2 derived in the first part.

If we try and find the above price by computing the expectation through a Monte Carlo simulation, we have to simulate both the behavior of S_t for the equity "C" underlying the swap, and the default of the counterparty "B". In particular we need to know exactly $\tau = \tau_B$. Obviously the correlation between "B" and "C" could have a relevant impact on the contract value. Here the structural model can be helpful: Suppose to calibrate the underlying process V to CDS's for name "B", finding the appropriate default barrier and volatilities according to the procedure outlined earlier in this paper with the AT1P model. We could set a correlation between the processes V_t^B (firm value for "B") and S_t^C (equity for "C"), derived for example through historical estimation directly based on equity returns, and simulate the joint evolution of $[V_t^B, S_t]$. As a proxy of the correlation between these two quantities we may consider the correlation between S_t^B and S_t^C, i.e. between equities.

Going back to our equity swap, now it is possible to run the Monte Carlo simulation, looking for the spread X that makes the contract fair.

We performed some simulations under different assumptions on the correlation between "B" and "C". We considered five cases: $\rho = -1$, $\rho = -0.2$, $\rho = 0$, $\rho = 0.5$ and $\rho = 1$. In Table 8 we present the results of the simulation, together with the error given by one standard deviation (Monte Carlo standard error). For counterparty "B" we used the Vodafone CDS rates seen earlier. For the reference stock "C" we used a hypothetical stock with initial price $S_0 = 20$, volatility $\sigma = 20\%$ and constant dividend yield $q = 0.80\%$. The contract has maturity $T = 5y$ and the settlement of the LIBOR rate has a semi-annual frequency. Finally, we included a recovery rate $\text{R}_{\text{EC}} = 0.4$ in $\text{L}_{\text{GD}} = 1 - \text{R}_{\text{EC}}$. The starting date is the same we used for the calibration, i.e. March 10th, 2004. Since the reference number of stocks K is just a constant multiplying the whole payoff, without losing generality we set it equal to one.

In order to reduce the errors of the simulations, we have adopted a variance reduction technique using the default indicator (whose expected value is the known default probability) as a control variate. In particular we have used the default indicator $1_{\{\tau \le T\}}$ at the maturity T of the contract, which has a large correlation with the final payoff. Even so, a large number of scenarios is needed to obtain errors with a lower order of magnitude than X. In our simulations we have used $N = 2000000$.

We notice that X increases together with ρ, and in Brigo and Tarenghi (2004) we explain why this is natural.

ρ	X	ERS payoff (basis points)	MC error (bps)
-1	0	0	0
-0.2	2.45	-0.02	1.71
0	4.87	-0.90	2.32
0.5	14.2	-0.53	2.71
1	24.4	-0.34	0.72

Table 8: Spread X (in bps) under five correlation values, $S_0 = 20$, basic AT1P model. We also report the value of the average of the simulated payoff (times 10000) across the 2000000 scenarios and its standard error, thus showing that X is fair (leads to a zero NPV).

To check the impact of the scenarios barrier, we have re-priced with the same X's found in Table 8 for AT1P our equity swap under the SVBAT1P model calibrated to the same CDS data, with the weighted calibration given in Table 6. If we consider the case with $\rho = 0.5$, the Monte Carlo method gives us the payoff expected value as 292.03 bps, with a Monte Carlo error of about 1.67 bps. Again, recalling that $S_0 = 20$ we can consider $292.03/20 \approx 14.6$, i.e. the price for notional unit, and compare with CDS payoff bid ask values, as in Table 2. We see that we are within the 7y and 10y CDS bid ask spreads but not within the 1y, 3y and 5y spreads. We try and see which value for X in the AT1P model with $\rho = 0.5$ would produce an expected payoff close to 290. We obtain that a spread of $X = 17.3$ would give an expected payoff value of 289, so that we see that the difference between AT1P and SVBAT1P in terms of AT1P spread is of about 17.3-14.2= 3.1 bps. By comparing with bid-ask spreads in CDS rates R, as given in Table 1, we see that this difference is inside the 1y, 7y and 10y spreads on the R quotes (which are respectively of 5, 8 an 10 bps). Also, even if a little larger, it is comparable to the 3y and 5y spreads (both of 2 bps), which are very narrow being related to the most traded CDS maturities. Thus we see that the difference is nearly negligible.

Since AT1P and SVBAT1P are calibrated to the same CDS data up to five years (but SVBAT1P is also calibrated to 7y and 10y CDS), we are seeing here that the different dynamics assumptions in the two models lead to different counterparty risk valuations in the equity return swap. The difference is not large when compared to bid ask spreads of CDS. We may expect more significant deviations in hedging. We have to keep in mind an important consideration, though. SVBAT1P is calibrated not exactly and not only on 1y, 3y and 5y CDS as the earlier AT1P model. Probably some of the difference between the price obtained with the SVBAT1P is to be attributed to this fact.

A final remark concerns the "nested calibration" of our models. With AT1P the calibration is "nested", in that adding one CDS with a larger maturity does not change the earlier σ parameters found with the calibration up to that point. In a way, this is a "cascade calibration". This may be helpful with sensitivities and bucketing. Instead, in the SVBAT1P model the parameters assume a "global" role: if we add a CDS quote and recalibrate, all the parameters change again. This is less desirable in computing sensitivities to market inputs and can lead to numerical problems. We will address these matters in future work.

11.1 Conclusions on Part III

In general the link between default probabilities and credit spreads is best described by intensity models. Yet, intensity models present some drawbacks: They do not link the default event to the economy but rather to an exogenous jump process whose jump component remains economically unexplained. Further, modelling correlation or more generally dependence between the contract underlying and the counterparty default may be difficult in this kind of models, given independence of the jump component from all default-free market quantities.

In this Part III we introduced analytically tractable structural first passage models based on a time varying volatility and default barrier (AT1P) and on scenarios on the value of the firm volatility and of the default barrier (SVBAT1P). These models allow for a solution to the above points. In these models the default has an economic cause, in that it is caused by the value of the firm hitting the default safety barrier value, and all quantities have a clear economic interpretation. Also, the model allows for the introduction of the correlation between counterparty and underlying in a very natural way, by simply correlating shocks in the value of the firm of the counterparty to shocks in the underlying.

We showed how to calibrate the models parameters to actual market CDS data, obtaining in general exact calibration in the AT1P model, and more realistic but slightly inexact calibration outputs with SVBAT1P.

Finally, the model can be used to build a relationship between the firm value V and the firm equity S (perceived as a suitable barrier payoff in terms of V itself), for example along the lines of Jones et al (1984) and Hull, Nelken and White (2004). This approach can be followed to price an equity default swap. We need to find an expression for the debt (and thus the equity) within the chosen structural model. Debt and equity expressions are known in closed form for time-constant and standard (exponential) barrier Black Cox models (see for example Bielecki and Rutkowski (2001), Chapter 3). Under SVBAT1P we simply obtain a linear combination of said formulas and we have a closed form expression for the equity. Then we can price an equity default swap by means of Monte Carlo simulation of the firm value, from which, scenario by scenario, we deduce analytically the equity value by means of the found formula. This is currently under investigation.

12 Conclusions

In this Chapter we considered a first approach to counterparty risk pricing when no collateral is given as guarantee. The price for this risk is computed in a risk neutral valuation framework. We gave an example of the impact of netting agreements in the swap example, while for some first considerations involving collateral we refer to Cherubini (2005). Also, we plan to analyze the impact of credit spread volatility (stochastic intensity) and of intensity/interest rate correlation on the swaption-counterparty risk.

In general further research will have to focus on hedging strategies associated with the prices we presented, and we plan to address this issue in further work.

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