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Credit Risk Evaluation

Modeling – Analysis – Management

by

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Uwe Wehrspohn

Introduction

In the 1990ies, credit risk has become the major concern of risk managers in financial institutions and of regulators. This has various reasons:

- Although market risk is much better researched, the larger part of banks' economic capital is generally used for credit risk. The sophistication of traditional standard methods of measurement, analysis, and management of credit risk might, therefore, not be in line with its significance.
- Triggered by the liberalization and integration of the European market, new channels of distribution through e-banking, financial disintermediation, and the entrance of insurance companies and investment funds in the market, the competitive pressure upon financial institutions has increased and led to decreasing credit margins[1]. At the same time, the number of bankruptcies of companies stagnated or increased[2] in most European countries, leading to a post-war record of insolvencies in 2001 in Germany[3].
- A great number of insolvencies and restructuring activities of banks were influenced by prior bankruptcies of creditors. In the German market, prominent examples are the Bankgesellschaft Berlin (2001), the Gontard-MetallBank (2002), the Schmidtbank (2001), and many mergers among regional banks[4] to avoid insolvency or a shut down by regulatory authorities.

The thesis contributes to the evaluation and development of credit risk management methods. First, it offers an in-depth analysis of the well-known credit risk models Credit Metrics (JP Morgan), Credit Risk+ (Credit Suisse First Boston), Credit Portfolio View (McKinsey & Company) and the Vasicek-Kealhofer-model[5] (KMV Corporation). Second, we develop the Credit Risk Evaluation model[6] as an alternative risk model that overcomes a variety of deficiencies of the existing approaches. Third, we provide a series of new results about homogenous portfolios in Credit Metrics, the KMV model and the CRE model that allow to better

1 Bundesbank (2001).

2 Creditreform (2002), p. 4.

3 Creditreform (2002), p. 16.

4 Between 1993 and 2000 1,000 out of 2,800 Volks- und Raiffeisenbanken and 142 out of 717 savings banks ceased to exist in Germany (Bundesbank 2001, p. 59). All of them merged with other banks so that factual insolvency could be avoided in all cases. Note that shortage of regulatory capital in consequence of credit losses was not the reason for all of these mergers. Many of them were motivated to achieve cost reduction and were carried out for other reasons.

5 We refer to the Vasicek-Kealhofer-model also as the KMV model.

6 Credit Risk Evaluation model is a trademark of the Center for Risk & Evaluation GmbH & Co. KG, Heidelberg. We refer to the Credit Risk Evaluation model also as the CRE model.

understand and compare the models and to see the impact of modeling assumptions on the reported portfolio risk. Fourth, the thesis covers all methodological steps that are necessary to quantify, to analyze and to improve the credit risk and the risk adjusted return of a bank portfolio.

Conceptually, the work follows the risk management process that comprises three major aspects: the modeling process of the credit risk from the individual client to the portfolio (the qualitative aspect), the quantification of portfolio risk and risk contributions to portfolio risk as well as the analysis of portfolio risk structures (the quantitative aspect), and, finally, methods to improve portfolio risk and its risk adjusted profitability (the management aspect).

The modeling process

The modeling process includes the identification, mathematical description and estimation of influence factors on credit risk. On the level of the single client these are the definitions of default[7] and other credit events[8], the estimation of default probabilities[9], the calculation of credit exposures[10] and the estimation of losses given default[11]. On the portfolio level, dependencies and interactions of clients need to be modeled[12].

The assessment of the risk models is predominantly an analysis of the modeling decisions taken and of the estimation techniques applied. We show that all of the four models have considerable conceptual problems that may lead to an invalid estimation, analysis and pricing of portfolio risk.

In particular, we identify that the techniques applied for the estimation of default probabilities and related inputs cause systematic errors in Credit Risk+[13] and Credit Portfolio View[14] if certain very strict requirements on the amount of available data are not met even if model assumptions are assumed to hold. If data is sparse, both models are prone to underestimate default probabilities and in turn portfolio risk.

For Credit Metrics and the KMV model, it is shown that both models lead to correct results if they are correctly specified. The concept of dependence that is common to both models – called the normal correlation model – can easily be generalized by choosing a non-normal

7 See section I.A.
8 I.e. of rating transitions, see sections I.B.4, I.B.6.c)(4), I.B.7.
9 See section I.B.
10 Section I.C.
11 Section I.D.
12 See Section II.A.
13 Section I.B.5
14 Section I.B.6

distribution for joint asset returns. As one of the main results, we prove for homogenous portfolios that the normal correlation model is precisely the risk minimal among of all possible generalizations of this concept of dependence. This implies that even if the basic concept of dependence is correctly specified, Credit Metrics and the Vasicek-Kealhofer model systematically underestimate portfolio risk if there is any deviation from the normal distribution of asset returns.

Credit Risk+ has one special problem regarding the aggregation of portfolio risk[15]. It is the only model whose authors intend to avoid computer simulations to calculate portfolio risk and attain an analytical solution for the portfolio loss distribution. For this reason, the authors choose a Poisson approximation of the distribution of the number of defaulting credits in a portfolio segment. As a consequence each segment contains an infinite number of credits. This hidden assumption may lead to a significant overestimation of risk in small segments, e.g. when the segment of very large exposures in a bank portfolio is considered that is usually quite small. Thus, Credit Risk+ is particularly suited for very large and homogenous portfolios. However, at high percentiles, the reported portfolio losses even *always* exceed the total portfolio exposure.

With the Credit Risk Evaluation model, we present a risk model that avoids these pitfalls and integrates a comprehensive set of influence factors on an individual client's risk and on the portfolio risk. In particular, the CRE model captures influences on default probabilities and dependencies such as the level of country risk, business cycle effects, sector correlations and individual dependencies between clients. This leads to an unbiased and more realistic estimation of portfolio risk[16].

The CRE model also differs from the other models with respect to the architecture, which is modular in contrast to the monolithic design in other models. This means that the cornerstones of credit risk modeling such as the description of clients' default probabilities, exposures, losses given default, and dependencies are designed as building blocks that interact in certain ways, but the methods in each module can be exchanged and adjusted separately. This architecture has the advantage that, by choosing appropriate methods in each component, the overall model may be flexibly adapted to the type and quality of the available data and to the structure of the portfolio to be analyzed.

[15] Section II.A.3.a)

[16] Sections I.B.7 and II.A.2

For instance, if the portfolio is large and if sufficiently long histories of default data are available, business cycle effects on default probabilities can be assessed in the CRE model. Otherwise, more simple methods to estimate default can be applied such as long term averages of default frequencies etc. Similarly, country risk typically is one of the major drivers of portfolio risk of internationally operating banks. In turn, these banks should use a risk model that can capture its effect[17]. Regional banks, on the other hand, might not have any exposures on an international scale and, therefore, may well ignore country risk.

Moreover, an object-oriented software implementation of the model can directly follow its conceptual layout. Here, building blocks translate into classes and methods into routines within the classes. This makes it easy to adapt the software to the model and to integrate new methods.

It is worth noting that the CRE model contains Credit Metrics and the Vasicek-Kealhofer model as special cases, if methods in the modules are appropriately specified.

The presentation of our analyses and results follows the modular architecture of the CRE model. We go through the building blocks separately and only analyze the respective component of each model and, if necessary, the restrictions that the choice of a particular model in one building block imposes upon other components. This structure renders it possible to assess each method in each module individually and to avoid that errors accumulate or offset each other and make the resulting effect intransparent and difficult to apprehend.

Analysis of portfolio risk structures

After all components of a portfolio model are defined and all relevant input parameters are estimated, the next step in the credit risk management process entails the quantification of portfolio risk and of risk contributions to portfolio risk and the analysis of portfolio risk structures. This step is entirely based upon the portfolio loss distribution and on the concept of marginal risks. As they are based upon standardized model outputs, all methods to analyze risk structures are generally valid and independent of the underlying risk model.

We develop a general simulation based approach how the portfolio loss distribution and the expected loss, the standard deviation, the value at risk, and the shortfall as specific risk measures can be estimated and supply formulas for confidence intervals around the estimated risk measures and confidence bands around the loss distribution. We also show that the calculation

[17] Sections I.B.7.a)(1) and II.A.5.

of value at risk and shortfall may be subject to systematic estimation errors if the number of simulation runs is not sufficiently large with regard to the required confidence level.

The mere calculation of risk measures for the entire portfolio and portfolio segments is usually not sufficient in order to capture the complexity of real world portfolio structures and to localize segments where the risk manager has to take actions. This is due to the fact that different aspects of risk such as segments' losses given default, their risk contributions, risk adjusted returns etc. may lead to very different pictures of the portfolio structure and may also interact. I.e. a portfolio segment, that appears to be moderately risky if single aspects of risk are considered in isolation, can gain a high priority if various concepts of risk and return are evaluated in combination. For this reason, we give an example of a comprehensive portfolio analysis and the visualization of portfolio risk in a 'risk management cockpit'.

A complementary approach to improve portfolio quality that does not depend upon the actual portfolio composition is algorithmic portfolio optimization. We develop a method that minimizes portfolio shortfall under certain side-constraints such as the size of expected returns or non-negativity of exposures and give an example of an optimization and its effect upon portfolio composition and marginal risk contributions.

Risk management techniques

When portfolio risk is modeled, measured and decomposed, the risk manager may want to take action to adjust the portfolio along value at risk, shortfall and return considerations. On the level of the single client this can be done by adequate, risk adjusted pricing of new credits and the allocation of credit lines. On the portfolio level, the allocation of economic capital as well as the setting of risk, exposure and concentration limits, credit production guidelines, and credit derivatives can be used, for instance, to redirect the portfolio.

The thesis is organized as follows: In the first part, we discuss the credit risk management of a single client. This includes the modeling and estimation of clients' risk factors and mainly the risk adjusted pricing of financial products. In the second part, the focus is on the risk management of multiple clients. We begin with a detailed description and analysis of various concepts of dependence between clients. Subsequent sections deal with the quantification and analysis of portfolio risk and with risk management techniques.

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I. The credit risk of a single client

The credit risk of a single client is the basis of all subsequent risk analysis and of portfolio risk modeling. In this part, we provide a comprehensive account of single credit risk modeling.

To clarify the events considered as 'credit risk', we start with different definitions of default. We continue with an in-depth analysis of various methods to assess default probabilities. In particular, we show the properties and sometimes deficiencies of the estimation techniques and propose new modeling ideas and estimation methods. The next two sections discuss exposure concepts and models for the loss given default rates or recovery rates, respectively. Finally, as one of the most important risk management techniques at the level of a single client, we give and prove formulas for the risk adjusted pricing of bonds and European options.

A. Definitions of default

While legal definitions of default vary significantly, two main concepts of default can be distinguished.

The first concept is client orientated, i.e. the status of default is a state of a counterparty such as insolvency or bankruptcy. The major consequence of this definition is that all business done with the respective counterparty is affected simultaneously in the event of default. Thus, all transactions are fully dependent upon each other. It is impossible by definition that, say, two thirds of a counterparty's transactions default while the remaining third survives. The single contracts differ only in the loss conditional to default, but not in the fact of default. This strong interdependence of the trades implies, firstly, that it is sufficient to trace the client's credit quality and financial prospects to assess the probability of default of each individual contract and, secondly, it allows the aggregation of credit exposures[18] from the single trades to a total exposure of the client as the relevant input to further risk management techniques.

The aggregation of exposures belonging to the same counterparty is not only useful in exposure limitation, but it is an important simplification in all simulation based portfolio models because it reduces the number of exposures to be modeled to the number of clients. Taking into consideration that the consumption of computer resources and calculation time increases

[18] See section I.C below.

quadratically in the number of exposures in most models, it becomes evident that it is an advantage to be able to keep the number of exposures small.

This client-orientated definition of default is adequate for derivatives and trading portfolios and most classical credits.

The second concept of default is transaction orientated. It is, thus, the direct opposite to the first approach. Here, default occurs if a contract is given notice to terminate[19]. This definition of default is particularly suitable if financed objects belong to the same investors, but are juridically independent. In this context it would not necessarily be justified to assume that all contracts default simultaneously[20]. Another application is joint ventures by a number of counterparties. In this situation it is hardly possible to assign the contract to a single client or to give a precise reason for default as all participants are liable.

All estimation techniques of default probabilities subsequently described can be used with the first definition of default that puts the focus on the client. The market data based approaches, however, are specialized on the calculation of individual default probabilities. Hence, they cannot be applied if the transaction-orientated concept of default is required.

All rating based techniques to estimate default probabilities can be used with both concepts. It is worth noting, though, that the Credit Risk Evaluation model, in particular, is designed to handle both concepts simultaneously. It can apply different methods to calculate default probabilities in parallel and can picture specific dependencies between clients and objects[21].

B. Estimation of default probabilities

1. Market factor based estimation of default probabilities: the Merton model

To determine whether a company has the ability to generate sufficient cash flow to service its debt obligations the focus of traditional credit analysis has been on the company's fundamental accounting information. Evaluation of the industrial environment, investment plans, balance sheet data and management skills serve as primary input for the assessment of the com-

[19] Sometimes default on a transaction level is defined as the event that a due payment is delayed. This definition has a problematic implication. Here default is no absorbing state any more as an insolvency or a notice to terminate. The delayed payment can be made later and the contract survives. Thus, there will be a cluster of observed recoveries after default at 100% since no loss actually occurs if the contract is carried on, whereas in other circumstances, where the delay is due to a more severe credit event, consequences are much more serious.

[20] It is, for instance, often observed that residential mortgage loans default up to 10-20 times less frequently than non-residential mortgage loans.

[21] See section I.B.7 below.

panies likelihood of survival over a certain time horizon or over the life of the outstanding liabilities.

It is a well-known critique of this approach that financial statement analysis may present a flawed picture of a firm's true financial condition and future prospects. Accounting principles are predominantly backward oriented and conservative in design. Moreover, accounting information does, therefore, not include a precise concept of future uncertainty. "Creative accounting" might even intend to disguise the firm's factual situation within certain legal limits. Finally, a market valuation a firm's assets is difficult in the absence of actual market related information.

In his seminal article on credit risk management[22] Robert Merton proposes a method to price a public company's debt based on the equilibrium theory of option pricing by Black and Scholes[23] [24]. Supposing that his arguments hold, some of the results can serve as an important input for the calculation of default probabilities.[25]

a) Main concept

Under the simplifying assumption[26] that all of the company's liabilities are zero-bonds with the same maturity, Merton defines the default of the company as being equivalent to the event that the total value of the firm's assets is inferior to its obligations at the moment of their maturity. In this case the owners would hand the firm over to the creditors rather than paying back the debt. The probability of default is, thus, equal to the probability of observing this event.

b) Assumptions

In order to be able to close the model[27] and to derive formulas, Merton makes a number of technical and fundamental assumptions. The technical assumptions serve above all to facili-

22 Robert Merton (1974)

23 Fisher Black and Myron Scholes (1973)

24 This is why Merton's model is often referred to as the option pricing approach.

25 The calculation of default probabilities was not proposed by Merton himself in the article quoted above. It is rather an extension of Merton's original approach, which has been initiated by KMV Corporation, San Francisco (from now on referred to as "KMV"). For further modifications of Merton's model by KMV see next chapter.

26 See below.

27 It is a central challenge for the model to deduce the hidden variables. Merton's interest was to price risky debt. All that can be concluded from his analyses are the input variables relevant for this purpose. The assumptions made – both technical and fundamental - have to be seen on that background.

For the calculation of default probabilities one further input is needed (the expected return on firm value) whose derivation remains a major concern for the practicability of the model.

tate the mathematical presentation and to obtain a tractable formalism and can be considerably weakened[28]. They are:

1. The market is "perfect" (i.e. there are no transaction costs nor taxes; all assets can be infinitely divided; any investor believes that he can buy and sell an arbitrary amount of assets at the market price; borrowing and lending can be done at the same rate of interest; short-sales of all assets are allowed).
2. The risk free rate is constant[29].
3. The company has only two classes of claims: (1) a single and homogeneous class of debt, precisely of zero-bonds all with the same seniority[30] and maturity. (2) The residual claim, equity.
4. The firm is not allowed to issue any new (senior)[31] debt nor pay cash dividends nor repurchase shares prior to the maturity of the debt.[32]
5. The Modigliani-Miller theorem obtains, i.e. the value of the firm is invariant to its capital structure.[33]

The fundamental assumptions are:

6. The value of the firm, V, follows an autoregressive process, i.e. all information needed to predict the future dynamics of the firm value is contained in its past development. The value of the firm is, particularly, not subject to any exogenous shocks. The assumption that V specifically follows a geometric Brownian motion

$$dV = \mu V dt + \sigma V dz$$

is again simplifying[34]. It implies that the volatility of the returns on firm value is constant over time and that the distribution of its growth rates is normal.

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28 Robert Merton (1974), p.450

29 If interest rates are constant there are no term structure effects so that term structure and risk structure effects on the price of debt and the probability of default can trivially be separated.

30 For the calculation of default probabilities, it is not necessary that all bonds have the same seniority since equity is always the most junior claim. It is just relevant for the pricing because a credit's expected loss usually depends on its seniority.

31 See footnote 30.

32 It would be sufficient that the nominal amounts of debt and of dividend payments were deterministic functions of time.

33 Merton himself shows that the argument holds even without the Modigliani-Miller theorem. However, this case is formally more complex because it leads to non-linear stochastic differential equations. See Merton (1974), p. 460.

34 Here μ is the instantaneous rate of return on the firm per unit time, σ is the instantaneous standard deviation of the return on the firm per unit time; *dz* is a standard Brownian motion.

7. The value of the firm's equity, *E*, (and, hence, debt) is a deterministic function of the value of the firm and time:

$$E = F(V,t)$$

Thus, by Itô's Lemma, the stochastic differential equation defining the distribution of *E* is explicitly given as

$$dE = \mu_E E dt + \sigma_E E dz_E$$

where σ_E, μ_E and dz_E are known functions of *V*, *t*, σ, μ and *dz*. It is essentially implied that the dynamics of the equity markets are fully induced by the stochastic behavior of asset values and that there is no further source of uncertainty in the equity markets as for instance by speculation or imperfect aggregation of information.

8. As a necessary condition for the previous supposition to hold[35] and as to be able to use the risk-neutral valuation argument by Black and Scholes to eliminate μ from the stochastic differential equation defining the behavior of *E*, Merton assumes that an ideal fully self-financed portfolio consisting of the firm, equity, and risk-less debt can be constructed and be valued using a no arbitrage argument.

9. Trading in assets takes place continuously in time so that the mentioned portfolio can be hedged at each point in time.

10. Total equity value is exactly the sum of all incremental equity values.

c) Derivation of default probability

From the definition of default as the event that the firm value is inferior to the total amount of debt at maturity of the debt and from the assumption that firm value follows a geometric Brownian motion, it would be straight forward to calculate the company's default probability if the so far hidden variables μ, σ, and V_0 were known.

It follows from the distribution of the stochastic process of *V* that the logarithm of firm value at time *T* is normally distributed with mean[36]

$$\mathbf{E}(\ln V_T) = \ln V_0 + (\mu - 0.5\,\sigma^2)\,T$$

and variance

$$\mathrm{Var}\,(\ln V_T) = \sigma^2\,T.$$

[35] If arbitrage were possible in the market, the price of equity would also depend on the size of the arbitrage opportunity. In this case, Itô's Lemma would be invalid. For Itô's Lemma see Øksendal (1998), Theorem 4.1.2.

[36] We assume that the present moment is equal to *t = 0*.

Therefore, we have[37]

$$\begin{aligned}&\mathbf{P}\{\text{default}\} = \mathbf{P}\{V_T < D\} = \mathbf{P}\{\ln V_T < \ln D\}\\ &= \mathbf{P}\left\{\frac{\ln V_T - \left(\ln V_0 + \left(\mu - 0.5\sigma^2\right)T\right)}{\sigma\sqrt{T}} < \frac{\ln D - \left(\ln V_0 + \left(\mu - 0.5\sigma^2\right)T\right)}{\sigma\sqrt{T}}\right\}\\ &= \Phi\left(\frac{\ln D - \left(\ln V_0 + \left(\mu - 0.5\sigma^2\right)T\right)}{\sigma\sqrt{T}}\right) = \Phi\left(-d_2^*\right)\end{aligned}$$

The remaining task is to assign values to μ, σ, and V_0.

From Merton's analysis σ and V_0 can be concluded:

From the above assumptions[38] it can be shown that μ drops out of the stochastic differential equation defining equity value, *E*. Hence, μ cannot influence equity value, i.e. *E* is independent of investors' risk preferences. Thus, any risk preferences that seem suitable can be assumed without changing the result. It is particularly simplifying to consider investors as risk-neutral implying that μ is equal to the risk-free rate *r* and that the discount factor for the risky investment is equal to e^{-rT}.

Furthermore, observing that at maturity, *T*, of the debt equity value is equal to

$$E_T = \max(0, V_T\text{-}D),^{39}$$

the stochastic differential equation defining the distribution of *E* can be solved using analogous arguments to Black and Scholes (1973) [40]

$$E_0 = V_0\,\Phi(d_1) - e^{-rT}\,D\,\Phi(d_2).$$

As already alluded, it is implied by Itô's Lemma[41] that

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37 With

$$d_2^* = -\frac{\ln D - \left(\ln V_0 + \left(\mu - 0.5\sigma^2\right)T\right)}{\sigma\sqrt{T}} = \frac{\left(\ln V_0 + \left(\mu - 0.5\sigma^2\right)T\right) - \ln D}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{V_0}{D}\right) + \left(\mu - \sigma^2/2\right)T}{\sigma\sqrt{T}}$$

d_2^* states how many standard deviations the expected value of $\ln(V_T)$ is away from the default point $\ln(D)$ and is, therefore, often named 'distance to default'.

38 Especially assumption 8.

39 Since if $V_T \geq D$ (with *D* = total debt), equity holders pay back the debt, and if $V_T \leq D$, equity holders hand over the company. Equity has, thus, the same cash flow profile as a European call option with maturity *T* and strike price *D*.

40 With

$$d_1 = \frac{\ln\left(\frac{V_0}{D}\right) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

41 The stochastic differential equation defining the distribution of equity value *E = F(V,t)* is given by

$$dF = \left(\frac{\partial F}{\partial V}rV + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial V^2}\sigma^2 V^2\right)dt + \frac{\partial F}{\partial V}\sigma V dz$$

$$\sigma_0^E E_0 = V_0 \Phi(d_1)\sigma .$$

Both equations can simultaneously be solved numerically for V_0 and σ.

It should be pointed out, however, that even if all stated assumptions are fulfilled, the stochastic process defining the equity value E is apparently[42] heteroskedastic, i.e. equity volatility σ_E is not constant but changes over time. It can, therefore, not be taken for granted that σ_E is known or that it can be estimated by the 30-day-volatility, which is normally used in the Black-Scholes model[43]. Therefore, σ and V appear to remain to a certain degree hidden fundamental variables that prevent the model from being fully closed.

This is far more true for the expected return on firm value, μ. Other than in the capital asset pricing model (CAPM), it cannot directly be calculated from market returns, but has to be estimated indirectly from the previously estimated firm value process. It does seem unlikely that this procedure still leads to very precise results.

d) Discussion

If correctly specified, the Merton model is able to compensate for a number of deficiencies of traditional credit analysis.

It provides a methodology to effectively include the market's perception of a company into credit analysis. The information contained in equity markets is inherently future oriented and, therefore, particularly valuable. Leading to a formula for a company's default probability, the model has a clear-cut concept of uncertainty that can serve as input to further credit analysis.

Furthermore, default probabilities can be individually assessed on a day-to-day basis for each public company. I.e. companies' risk profiles can, firstly, be evaluated without a long time lag[44] so that possible deteriorations in credit qualities can be quickly anticipated and, secondly, risk profiles can be compared on a cardinal scale rather than just on an ordinal scale.

The equity volatility follows from the second term where $\frac{\partial F}{\partial V}$ is the option delta $\Phi(d_1)$. Cf. for Itô's Lemma to Øksendal (1998), Theorem 4.1.2.

[42] See above.

[43] In the Black-Scholes model the equity value is assumed to be homoskedastic, i.e. have constant volatility. An applet that illustrates the relationship between firm value, debt and default probabilities in the Merton model and also the heteroskedasticity of the equity price process which induces the mentioned estimation problems is available at http://www.risk-and-evaluation.com/Animation/Merton_Modell.html.

[44] This is not necessarily true because the amount of debt drawn by a company is not published on a day-to-day basis but rather parallel to the accounting periods. There might also be unknown undrawn lines of credit that could in reality be used to honor payments and avert a default.

There is no estimation error due to averaging between firms as in the rating based approaches (see below). Since the estimation is merely an evaluation of a stochastic process whose general distribution is known by assumption, there is no sampling error involved in the estimation. Having completely tied down default analysis to the purely quantitative analysis of the firm value process, all errors attributable to judgmental analyses by credit experts are avoided.

Besides the fact that input variables might not be fully known, the main points of criticism concern the validity of the fundamental assumptions made in the model.

The concept of default that a company goes bankrupt if and only if the total amount of assets is inferior to the total amount of debt at a certain point in time does explicitly exclude other reasons of insolvency frequently observed such as temporary liquidity problems, law suits, criminal acts etc. This narrow definition might lead to a misestimation of the probability of default.

It is particularly problematical that in order to be able to apply Itô's Lemma Merton assumes equity value to be a deterministic function of only asset value and time[45]. Although surely strongly influenced by a company's fundamental economic facts, it is largely uncontested that equity values are superimposed by speculative tendencies[46] and market imperfections that lead to inefficient aggregation of information.[47] Sobehart and Keenan (1999) show that default probabilities are overestimated if the fraction of equity volatility induced by asset volatility is overstated[48].

The requirement is also violated if a company's traded equity is not highly liquid so that the noted price might not be identical to the asset's actual market price. This quite restricts the number of companies accessible to analysis even among the public companies.

The supposition that the firm value follows an autoregressive process restricts the assessment of a company's creditworthiness solely upon the performance of its stock price. Exogenous influences such as country risk, fluctuations in the economic environment, business cycle effects, and productivity shocks that might change the characteristics of the firm value process are systematically ignored. It becomes clear from this fact that Merton's model is not an extension or a generalization of traditional credit analysis but rather disconnected from it. This

[45] It is worth noting that Merton apparently assumes this deterministic relationship for 'fundamental' technical reasons rather than for its economic realism. For the suppositions necessary for the validity of Itô's Lemma confer to Øksendal (1998), Theorem 4.1.2.

[46] Equity prices may even contain a bubble component. This is obvious given the recent experience with internet stocks. See Money Magazine April 1999, p. 169 for Yahoo's P/E ratio of 1176.6.

[47] For a detailed discussion of possible influences on equity values confer to Lipponer (2000), p. 66-69.

[48] Sobehart and Keenan (1999), p. 22ff. The authors state this fact as a major reason why equity and bond markets lead to very inconsistent results when it comes to credit analysis.

observation is so much more important as imprecisions in distributional assumptions or data quality are necessarily carried through to the estimation of default probabilities if there is no plausibility check against other economic variables that contain similar information.[49]

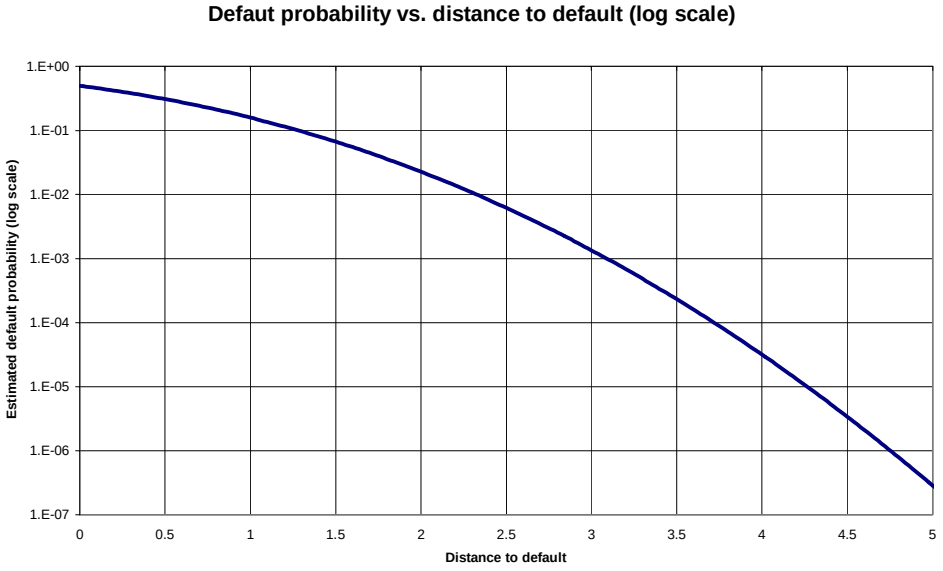

Figure 1: Default probability vs. distance to default in Merton's model

Figure 1 shows the "distance to default"[50], plotted against the corresponding probability of default[51]. It can be seen that the relationship between both variables is almost exponential implying that small misconceptions of the distance to default can lead to considerable mises-timations of default probability. A holistic credit analysis would, thus, try to combine market and accounting related information, if available, to increase precision to a maximum.

Merton's approach to deduce firm value, *V*, and firm value volatility, σ, would not be justified if the differential equation defining the equity value involved the expected return on the firm value, μ, since μ cannot be directly estimated and is not independent of investors' (unobserv-able) risk preferences. The higher the level of risk aversion by investors, the higher μ will be for any given firm. Merton, therefore, assumes the existence of arbitrageurs who imply that the self-financed portfolio consisting of the firm, equity, and risk-less debt earns the same risk-free rate as other risk-free securities, independent of μ. If the portfolio earned more than this return, arbitrageurs could make a risk-less profit by shorting the risk-free securities and

[49] This raises the question where lenders should take position in the trade-off between the error owing to the purely quantitative Merton approach and the error as a result of biased judgmental analyses of a company's 'soft facts'.

[50] See footnote 37.

[51] See the formula above.

using the proceeds to buy the portfolio; if it earned less, they could make a risk-less profit by doing the opposite, i.e. by shorting the portfolio and buying risk-free securities.[52]

It is, however, doubtful to take the existence of arbitrageurs for granted in a credit risk context. If equity is a long call option on firm value, the firm's creditors can be considered to hold the short position of the same option. The situation of creditors and equity owners in this case is slightly different from the short and long position of an ordinary call option, though, because equity holders remain the owners and the managers of the firm in addition to their long option position. This gives them the opportunity to dispose relatively freely of the firm's assets as it serves their interests. Creditors, on the other hand, as holders of the short position can do little to prevent this until after a default has occurred.[53] Creditors are, thus, in a much weaker position than holders of short positions in an ordinary context and, having to face this moral hazard problem of their counterparties, are not necessarily ready to function as arbitrageurs.

Finally, the equity price stated in financial markets is the price for one share only. It is evident from many take-over attempts that a company's total equity value can be very different from the sum of all incremental equity values.[54] Hence, total equity value can be considered as another hidden fundamental variable in the model.

2. Extensions of Merton's model by KMV

The many technical deficiencies of the Merton model greatly diminished its practicability for banks and lenders who wished to assess the default probability of their counterparties. This led KMV Corporation in the early 1980's to extend the Merton model to a variant, the Vasicek-Kealhofer model.[55]

The Vasicek-Kealhofer model[56] has the same conceptual architecture as the Merton approach, but above all tries to weaken and to adapt the technical assumptions[57].

[52] By this argument, Merton tries again to draw an analogy to the Black-Scholes analysis.

[53] Confer to Sobehart and Keenan (1999), p. 19f. See also footnote 48.

[54] For instance the stock value of Mannesmann increased by 100 billion DEM or more than 100% between October 1999 and February 2000 during an unfriendly takeover by Vodafone.

[55] Confer to Vasicek (1984).

[56] For clarity, we will refer to this model either as the Vasicek-Kealhofer model or the KMV model.

[57] Confer to Vasicek (1984), Crosby (1997), Crouhy et al. (2000), and Sellers et al. (2000).

While Merton assumes the firm's liabilities to only consist of two classes, a debt issue maturing on a specific date and equity, KMV allows liabilities to include current liabilities, short-term debt, long-term debt, convertible debt, preferred stock, convertible preferred stock, and common equity.[58]

KMV takes account of dividend payments and cash payments of interest prior to the maturity of the debt[59].

KMV generalizes the concept of default. In the Merton model, default was equivalent to the firm value being lower than the debt at the moment when the debt had to be repaid. In the Vasicek-Kealhofer model default can happen even before the maturity of a particular debt issue.[60] Equity, in this context, has no expiration date, but is modeled as a perpetual option.[61]

In the KMV model, the firm value process is only modeled as a geometric Brownian motion for the purpose of calculation of the unknown input variables and the distance to default[62].[63] It had turned out that the mapping of the distance to default measure to default probabilities via the lognormal law implied by the geometric Brownian motion led to implausible results.[64] Today an empirical distribution is used to assign default probabilities to the stated distances to default.

a) Discussion

The KMV model is valuable because it rendered the Merton model operational and turned it into a useful tool for practitioners. Although specialized on public companies[65], this is a group of counterparties that contributes particularly high credit risk to most banks' total portfolio due to small headcount and large volumes.[66]

Very importantly, the KMV model enables risk managers to monitor public companies on a day-to-day basis and use the estimated default probability[67] as early warning information that

58 See for instance Vasicek (1984), p. 5 and 11.

59 Vasicek (1984), p. 6 and 11.

60 Vasicek (1984), p. 5f.

61 Sellers et al. (2000), p. 3.

62 See footnote 37.

63 See Crouhy et al. (2000)

64 See Sellers et al. (2000), p.3, where it is stated that the normal law assigned a AAA rating to half of the North American companies in the KMV database.

65 Around 9500 companies in the U.S., see Sellers et al. (2000), p. 3.

66 This is particularly true for large, internationally operating banks.

67 KMV calls the output of its model an "expected default frequency" or "EDF".

is entirely based on automated and purely quantitative analysis. Once in operation, the model is unlikely to fail due to human misinterpretation of the actual economic situation.[68]

However, being an extension of the Merton model, the KMV model inherits all its severe structural problems.[69] Moreover, KMV has so far refused to publish the precise methodology and the data upon which the empirical distributions are based meaning that the model can be viewed as the proverbial black box.

This cannot be compensated by the fact that KMV asserts to have done detailed research that has proved all results.[70] The lack of a publicly available test is critical because the relationship between distance to default and estimated probability of default is so sensitive that small errors in the measuring of the distance to default or in the mapping between both quantities may lead to significant errors in the resulting default probability.[71]

3. Market factor based estimation of default probabilities: the Jarrow-Turnbull models

In a series of articles[72] beginning in 1995, Robert Jarrow and Stuart Turnbull developed a number of models under various assumptions and degrees of complexity that based the understanding of a trade's credit risk on the analysis of credit spreads and other relevant market factors[73]. Similar to Merton, Jarrow and Turnbull are predominantly interested in the pricing of financial securities subject to credit risk. They do not put the focus on the calculation of actual default probabilities.

a) Main concept

While considerably differing in detail, all Jarrow-Turnbull-models have a similar architecture consisting of four building blocks:

1. A model for market factors that influence the size of credit spreads such as the term structure of default-free interest rates and an equity market index. The actual shape of the models varies with the larger context. However, these 'elementary' factors are

[68] In its public firm model, Moody's Investors Service identified the capability to act as an early warning system as a major goal for a rating system. Cf. Sobehart et al. (2000), p. 5.

[69] See above.

[70] For a remarkable example of KMV's marketing activities see Sellers et al. (2000).

[71] See Figure 1.

[72] Confer to Jarrow et al. (1995), (1997a), (1997b), (2000), Jarrow (2000).

[73] This technology is implemented by Kamakura Corporation, Honolulu, USA, as a commercial software package under the name of Kamakura Risk Manager-Credit Risk System.

generally modeled as correlated autoregressive processes such as general Itô-Processes[74], geometric Brownian motions or mean-reversion processes[75].

2. The client's default process. It is supposed to be a binomial process or an exponential process independent of the development of the previously defined market factors[76], or a Cox process where the intensity is a function of the level of interest rates and the unanticipated changes in the equity market index.

3. The recovery rate process. Recovery rates conditional to default are assumed to be a constant fraction of the bond's present value prior to default[77] or of the bond's legal claim value, i.e. its principal plus accrued interest[78]. Alternatively, recovery rates are modeled as endogenous variables and are estimated from equity and expected bond prices[79].

4. A model that relates observed credit spreads to expected bond-payoffs and the risk-free interest rate curve[80].

Postulating the absence of arbitrage opportunities and a frictionless market, a risk-neutral world can be assumed. This implies that expected returns and discount factors are equal for different investments and can directly be deduced from the default-free interest rate curve[81]. Under these conditions the models can be solved and risk-neutral default probabilities be derived.

From the martingale condition, risk-neutral default probabilities, and the market factors, natural default probabilities could be calculated. However, being merely interested in pricing securities, Jarrow and Turnbull only briefly hint at this possibility[82].

Note that Jarrow and Turnbull do not need a clear-cut definition of default as in the Merton model or in the rating based estimation techniques to derive default probabilities. They merely require that default is an absorbing state. A firm in default will not come back. Supposing that the market's perception of a firm's financial future and it's prospects of default are implied in its credit spread, it is not necessary for the risk manager to further monitor the firm or to give reasons under what conditions a default could occur.

74 E.g. Jarrow et al. (1995), p. 71; (1997a), p. 275.
75 Jarrow et al. (2000), p. 284.
76 Jarrow et al. (1995), pp. 58, 73; (1997a), p. 273.
77 Jarrow et al. (1995), pp. 58; (1997a), p. 275.
78 Jarrow et al. (2000), p. 288.
79 Jarrow (2000).
80 Jarrow et al. (2000), p. 290f. even introduce a convenience yield compensating for short sale constraints.
81 This is the same argument as in the Merton-model.
82 Jarrow et al. (1997a), p. 292.

b) Discussion

The Jarrow-Turnbull-approach is the most important family of models that can make use of background market factors such as interest rates or equity prices to calculate a client's default probability and his credit exposure at the same time and, thus, integrate market and credit risk to a certain extent. This is especially an advantage if large portfolios of interest rate sensitive products such as bonds, swaps, and interest rate options need to be valued and hedged.

Being entirely based upon market data, the derived natural probabilities of default can also serve as early warning information for deteriorations in clients' credit quality and the general stability of markets. In addition, the model output could be a valuable point of comparison to the results of the Merton-model that also employs market data to estimate default probabilities.

Besides the apparent fact that it can only be used for firms with publicly traded bonds, the approach shares a number of disadvantages with the Merton-model. These are above all the data constraints and the hidden fundamental variables.

Apart from depending on market and default risk, credit spreads are frequently contaminated by other disturbing influences such as liquidity problems and especially by recovery rate uncertainty. This is particularly crucial because whilst recovery data is certainly the least reliable in credit risk analysis at all it decisively influences both the estimated default probabilities and security prices[83].

The same holds true for liquidity shortages that occur often in bond trading.[84] What is more, combined with the observation that corporate bonds are usually traded in small quantities with large notionals rather than in large numbers and small notionals such as shares, liquidity problems indicate that the assumption that arbitrageurs exist and constantly adjust prices to their accurate level is doubtful.

Here again we have to conclude that it is not an easy task to base the estimation of default probabilities on market data. If the necessary data can be obtained at all, its quality is not assured. Assumptions are required to close the model that certainly cannot be taken for granted. Moreover, it is not evident and has been left to future research as to how robustly the model reacts if assumptions are violated.

[83] In most models by Jarrow and Turnbull, it can be shown that for a given credit spread the estimated default probability also goes to one when the recovery rate tends to one.

[84] This is also stated by Jarrow and Turnbull themselves for bonds traded abroad. Jarrow et al. (2000), p. 291.

4. Rating based estimation of default probabilities: the mean value model

The third important method to estimate clients' default probabilities is based upon ratings.

A rating, in its most general definition, is an evaluation of a counterparty's credit quality. It is, in particular, an assessment of a client's probability to fail to meet its obligations in accordance with agreed terms.[85] Ratings have been developed since the early 20th century from investors' need for more market transparency and independent benchmarking, and from companies' necessity to open access to capital markets and to reduce refinancing costs.

Ratings present a much broader approach to estimate default probabilities than the purely market data oriented concepts previously discussed. Ratings typically try to evaluate all information at hand about a client. For example

- the market data, if available,
- other existing ratings,
- company financial statement information,
- macroeconomic variables that reflect the state of the economy and the company's specific industry,
- 'soft facts' such as management quality.[86]

This flexibility, with respect to the possible data basis, is a key advantage of the rating methodology compared to market data based approaches since it allows rating agencies and financial institutions to include all counterparties into the analysis. These clients can include public companies such as small and middle sized professionals and even private customers.

a) Main concept

The rating analysis proceeds in three steps:

1. Evaluate all credit quality relevant information related to a customer.
2. As a result of this first investigation, assign each client to a 'risk group'. A risk group is a set of counterparties who are assumed to be homogeneous in terms of credit risk, i.e. they are presumed to have the same default probability and the same probability to migrate

[85] See Sobehart et al. (2000), p. 6, and Standard and Poor's internet site http://www.standardandpoors.com/ratings/frankfurt/was.htm.

[86] Cf. footnote 85.

from one risk group to another. From this point all individual features belonging to a client are neglected and he is fully reduced to being a member of that specific group.

The definition of a risk group, i.e. a set of clients considered to be homogeneous, is one criterion in which different models can be distinguished. It can be a rating category such as AAA, AA to D in Standard and Poor's notation. This is the case in the mean value model. It can also be a rating category in combination with the size of the customer as a large multinational company and a small professional or a private customer despite being essentially different in their default behavior may be assigned to the same rating category. Models that try to estimate a counterparty's default probability against the background of its macroeconomic environment usually even make a distinction between rating categories in different sectors and countries because companies with the same risk profile are not necessarily equally related to the macroeconomy.

3. Finally, default probabilities are estimated for each risk group and a certain time horizon, usually of a year. It is clear that the probability of a company to get into financial distress is dependent on the length of the period of time under consideration.

As a second class of output from the rating process, the probability of migrations between risk groups can be estimated on the same time horizon. Migrations are assumed to reflect credit quality changes which can lead to changes in credit exposure.[87]

b) Derivation of default probability

In the mean value model, the estimation of default and migration probabilities is conceptually very simple. It is assumed that a risk group's default probability equals its observed historical average default rate. It is also presumed that the default probability is constant over time, not influenced by the present position in the business cycle or long term changes in the general economic situation, and that defaults are serially independent[88].

It is worth noting that the mean value model, as any other rating based approach, is open with respect to the definition of default. Other than the Merton model where default is indirectly defined as the event when the firm value is lower than the debt value at a certain point in time meaning that the company is unable to meet its obligations, the mean value model can be directly related to any explicit concept of default such as bankruptcy or just a missed payment.

[87] Some rating agencies such as Moody's Investors Service also estimate volatilities of default probabilities as a third kind of output.

[88] I.e. defaults in one period are independent of defaults in another period.

This is possible because the mean value model does not try to explain or give any reasons why a default occurs. It merely states the fact and tries to find a statistical relationship between a counterparty's credit quality and its financial and economic situation. The statistical link between the client's economic position and its probability of distress is purely correlative in nature, i.e. there is no reason given for the event of default as in the Merton model[89], it just happens that certain features tend to appear simultaneously no matter why.

This is also the motive why counterparties are first assigned to a risk group before default and migration probabilities are estimated. Since there is no causative linkage implied among the included variables and the credit event, it is obligatory to observe historical default frequencies as an input for the estimations. If, however, each client is assessed individually, the object under consideration ceases[90] to exist if a default happens, and the estimations are obsolete. Thus, the method requires counterparties to be clustered to a group as an intermediate step allowing the group to remain in existence even after defaults have been observed and the relevant input gathered.

The opposite is also true. Since the diversity of possible reasons for default is remarkable, it is indeed impossible to precisely trace all potential influence factors and statistically model their relationships so that the event of default can be described for each counterparty. This is so much more the case if there is no comprehensive indicator of credit quality to which the problem can be reduced such as highly liquid equity prices or bond spreads. In order to render the assessment of a client's credit quality operational, it is, thus, essential to abandon the structural approach and replace it with a methodology that whilst leading to efficient results, is also easier to handle for financial institutions and rating agencies.

c) Discussion

Ratings allow non-public companies and private customers to be assessed for credit risk. While market data based approaches were always restricted to certain segments of counterparties, ratings allow a financial institution to consistently estimate the default probabilities of its entire pool of clients. This is the core advantage of the rating methodology.

Moreover, ratings permit the efficient use of information. If market data is available, it can be included in the analysis as is the case in Moody's Risk Calc model[91]. However, the analysis is

[89] This is why the Merton model is also called the structural approach, while the mean value model is sometimes referred to as an 'ad hoc model'.

[90] This is particularly true if default is defined as bankruptcy. Here it is excluded that counterparties come back after a default.

[91] Cf. Sobehart et al. (2000)

not limited to market data. All other accessible information can be taken into account to underpin or partially correct the results.

This advantage is also one of the major challenges of the rating methodology. As the relevant data tends to be rather heterogeneous it can sometimes be extremely difficult to assign a client to a specific risk group. While financial statement information can well be integrated using statistical discriminance techniques, this is especially so for so called soft facts including management quality, diversification within a company or a firm's competitive position in the industry. Their evaluation is predominantly based upon expert knowledge and, therefore, less objective and prone to misconceptions and misunderstandings.

Other possible imprecisions result from the estimation of default probabilities. It is clear that the assumption that all counterparties within the same risk group have the same probability of default can only be a rough simplifying approximation. It would be much more intuitive to suppose that credit qualities are continuously distributed rather than jump from one rating category[92] to the other. By (mis-) interpreting a rating category as homogeneous the better clients are unduly devalued and their credit risk is overstated and in turn overpriced. On the other hand, the risk implied by the lower portion of the clients is underestimated and, therefore, underpriced.

This general flaw in the rating theory raises arbitrage opportunities for financial institutions that are in the position to more accurately assess their clients' credit risk. They can offer slightly favorable conditions to those counterparties in a rating category who are better than the stated average and try to win them as clients. Conversely, they can offer the right price to those below the stated average and maybe lose them as clients, but not expectedly lose money.

This argument already shows that financial institutions should not distinguish too few rating categories in order to hold the error within acceptable bounds.[93] Figure 2 illustrates the impact of the estimation error caused by a small number of rating categories on a portfolio level.

[92] In this section, we will use the terms 'risk group' and 'rating category' synonymously.

[93] See the detailed discussion below.

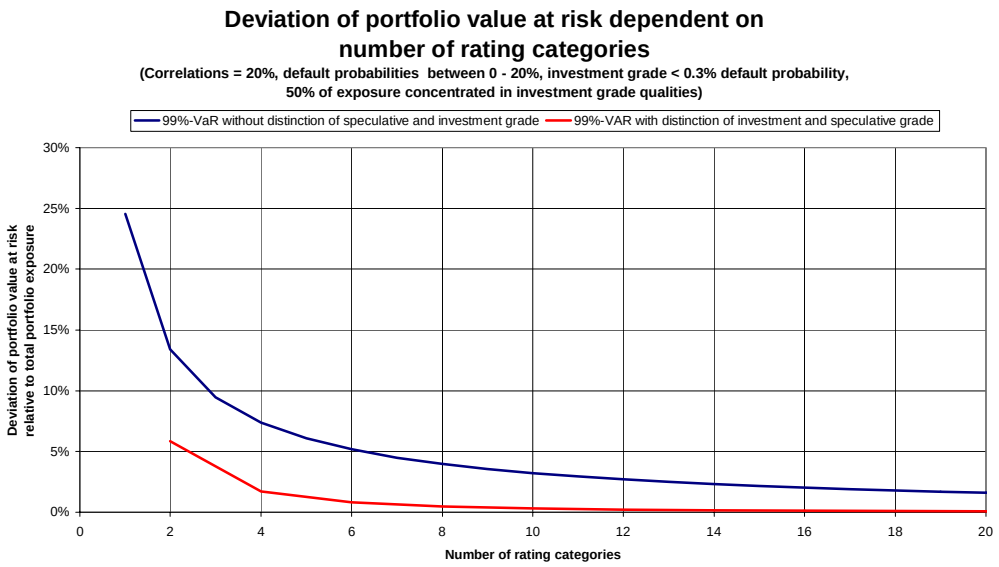

Figure 2: Deviation of portfolio value at risk dependent on number of rating categories

It shows the deviation of the value at risk[94] of the credit loss distribution of a portfolio where the counterparties' default probabilities are uniformly distributed between 0 and 20%[95]. To make the picture more realistic, it was further assumed that exposures were very uneven, i.e. that the amount of credit each client had received was a function of its credit quality[96]. Excellent qualities received up to 600 times more credit than the lowest ones. Since financial institutions keep most of their assets in good credit qualities, 50% of the total portfolio value were concentrated in the investment grade including default probabilities of less than 0.3%. To be able to easily parameterize dependencies among counterparties, the normal correlation model was used with all correlations being set to 20%[97].

Two different concepts of rating categories were distinguished. The first divided the interval from 0 to 20% in *n* equal segments, considering the mean of each segment to be its default probability. Taking account of the unevenly distributed exposures, the second concept made a distinction between an investment grade for excellent credit qualities with a default probability of less than 0.3% and a speculative grade for the rest.

It can be clearly seen that the misestimation of the portfolio value at risk sharply decreases if the number of rating categories increases. This is due to the fact that the assumption of homogeneous rating categories in terms of default probability becomes less simplistic if there are

94 We define the value at risk as a percentile of the portfolio loss distribution, i.e. as the smallest loss that is not exceeded with a probability of, say, 99%.

95 To avoid simulation errors the portfolio was supposed to consist of an infinite number of counterparties. In this case the value at risk can be written as an integral with a simple numerical solution. For details see below section II.A.1.c).

96 The exposures were defined by two linear functions $ax+b$, x being the client's true probability of default, one for investment grade qualities making up for 50% of the total portfolio value and one for speculative grade qualities.

97 For an explanation of the normal correlation model, the portfolio approach derived from the Merton model, as known from Credit Metrics or KMV, see below section II.A.1.

more categories. If the number of rating categories tends to infinity, the deviation of the estimated value at risk from the true portfolio value at risk goes to zero.[98]

However, the practicability of a rating system deteriorates greatly if there are too many categories. It turns out that the estimation error can be considerably reduced also with a smaller number of different rating categories if an investment and a speculative grade are distinguished. Both grades are not so much defined by credit quality, but rather by exposure concentrations. Banks should be able to make subtler distinctions of credit quality in customer segments to which they have lent large amounts of money no matter how good the clients' quality is in absolute terms. Large exposures tend to have a strong impact on portfolio risk. Like a magnifying glass, they sharply intensify misestimations of default probabilities on the portfolio level. Exposure concentrations are, therefore, the place where an exact assessment of default probabilities most pays off. In our example, the deviation from the true value at risk could be reduced to practically zero with just 10 rating categories.

Financial institutions should also try to keep track of the clients' changes in credit quality. The assumption of homogeneity of rating categories becomes more and more distorted if an up or downgrade of a clients' rating remains undetected, e.g. before the routine check of the rating. This kind of misspecification in turn biases the estimated default and migration probabilities and leads to an unwanted overlap of rating categories.[99] In order to operate a rating methodology effectively, it is, therefore, essential to implement an efficient early warning system that, ahead of schedule, hints at clients whose credit quality is about to significantly change and who should be put on a special watch list.[100] [101]

It is, however, important to be aware of the fact that even if a risk group is perfectly homogeneous with a constant default probability, the observed historical default rates of this group are random. For example, take a rating category consisting of 100 identical and independent clients all with a default probability of 1%. In this situation the probability to observe exactly one default is approximately 37% while the probability to observe no default at all or more than one default is more than 63%.

Hence, being a function of those historical default rates, the estimated default probabilities[102] are random, too. Figure 3 shows simulated distributions of estimated default probabilities for

98 This is exactly what is stated by the theory of Riemann integrals. The result is intuitively clear because if the number of rating categories grows, the categories tend to the case of individually assigned default probabilities.

99 See Kealhofer et al. (1998), p.11f.

100 This is also an insight from the Asian crisis 1997-1998 where rating agencies failed to recognize the upcoming events.

101 See e.g. Sobehart et al. (2000).

102 The estimated default probabilities were said to be the average historical default rates in the mean value model.

different degrees of dependence between the clients[103].[104] The rating category was supposed to be composed of 1000 identical counterparties with a true fixed probability of default of 0.5%[105] and various degrees of dependence. Their default behavior was followed over 15 periods and the default probability estimated from the observed default rates.

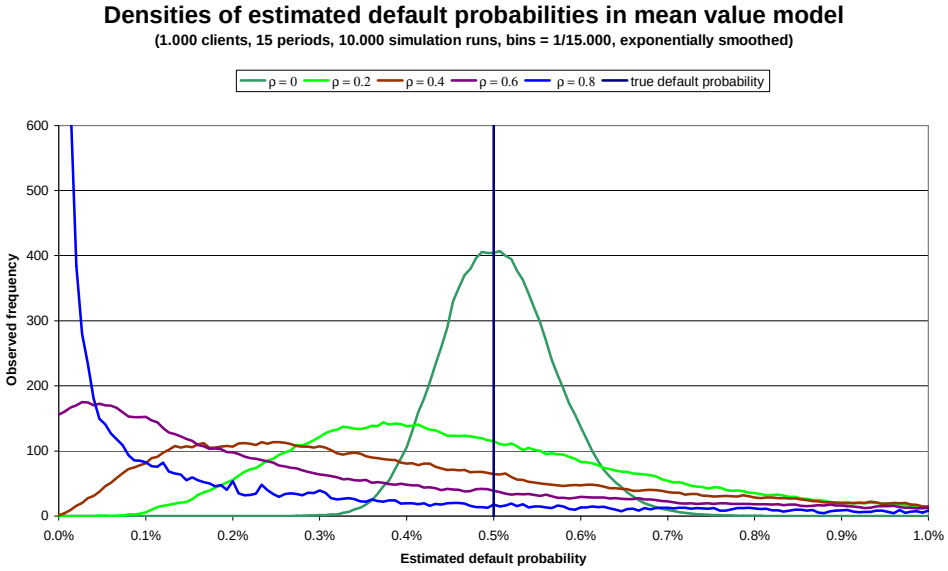

Figure 3: Distributions of estimated default probabilities in mean value model

Table 1 sums up some of the major characteristics of the distributions. The first striking fact is that the mean is always an unbiased estimator of default probability independent of the degree of dependence among the clients.[106]

	rho = 0	rho = 0.2	rho = 0.4	rho = 0.6	rho = 0.8	rho=0.9
Percentile of true value	50.00%	58.13%	63.66%	69.14%	77.29%	82.72%
Range 0[107]	0.29%	0.02%	0.00%	0.00%	0.00%	0.00%
5%- percentile	0.41%	0.21%	0.09%	0.02%	0.00%	0.00%
20%- percentile	0.45%	0.31%	0.18%	0.07%	0.01%	0.00%
40%- percentile	0.48%	0.40%	0.30%	0.17%	0.03%	0.00%
50%- percentile (median)	0.50%	0.45%	0.37%	0.25%	0.07%	0.01%
60%- percentile	0.51%	0.51%	0.46%	0.35%	0.14%	0.03%
80%- percentile	0.55%	0.66%	0.74%	0.77%	0.62%	0.36%
95%- percentile	0.60%	0.94%	1.37%	1.89%	2.83%	3.36%
Range 1	0.72%	2.49%	5.44%	10.11%	10.57%	12.79%
Std.Deviation	0.06%	0.23%	0.45%	0.71%	1.09%	1.27%
Mean	0.50%	0.50%	0.50%	0.51%	0.51%	0.48%

[103] The correlation parameter ρ again refers to risk index correlations in the normal correlation model, see below section II.A.1. Note that in this model uncorrelated counterparties are independent.

[104] For a similar analysis confer to Kealhofer et al. (1998), p. 7ff.

[105] The long-term mean default rate in the German economy between 1975 and 1992 was around 0.51%. Cf. to Bär (2000).

[106] This follows from the definition of the expected value. It is also independent of the number of observed periods and from the number of clients.

Skewness	2,10E-11	1,71E-08	2,22E-07	1,14E-06	4,63E-06	7,31E-06
Kurtosis	3,47E-13	2,01E-10	5,31E-09	4,55E-08	2,62E-07	4,38E-07

Table 1: Characteristics of simulated distributions of probability estimator in the mean value model

Conversely, the distributions' standard deviations, their skewness', kurtosis', and ranges, and the percentile represented by the true default probability do increase with the degree of dependence[108].

The case of independent clients is set off against the case of higher correlations because here the average default rate is normally distributed due to the central limit theorem. This is quite well indicated by the simulation results since the normal distribution is symmetric around the mean[109] with zero skewness and kurtosis. The instance of independence is the only one where the average default rate is already a consistent estimator of default probability when the number of clients tends to infinity[110].

If correlations are positive, the standard deviation of the average default rate remains positive even if the number of clients in this rating category is infinitely large, although it is decreasing in the number of clients. If, however, the number of periods goes to infinity, the default probability can be consistently estimated no matter how many clients there are.[111]

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107 Range 0 is the 0%-centile, i.e. the smallest simulated value of the average default rate. Equivalently, range 1 is the 100%-centile, i.e. the highest simulated value.

108 An animation that illustrates the dependence of the distribution of the mean default rate on the number of periods of data available and the correlations between clients is available at http://www.risk-and-evaluation.com/Animation/Mean_Value_Model.gif

109 This is why the mean and the median are identical.

110 This means that the average default rate converges stochastically against the true default probability in the number of clients.

Consistency implies that the precision of the estimation improves if the amount of available data increases, and that the sampling error diminishes.

111 It is straightforward to show that the standard deviation σ of the average default rate can be directly calculated as

$$\sigma(m,n,p,\rho_D) = \left(\frac{1}{m}\left(\frac{1}{n}p(1-p) + \left(1-\frac{1}{n}\right)\rho_D\, p(1-p)\right)\right)^{1/2}$$

where m is the number of periods observed, n is the number of clients, p the true probability of default and $\rho_D = \rho_D(\rho,p)$ is the default correlation. In the normal correlation model, ρ_D is given by

$$\rho_D = \frac{\Phi(a,a;\rho) - \Phi(a)\Phi(a)}{\Phi(a)(1-\Phi(a))}$$

with $\Phi(\cdot)$ being the one dimensional cumulative normal distribution function, $\Phi(\cdot,\cdot;\rho)$ the two dimensional cumulative normal distribution function with correlation ρ, and finally with $a = \Phi^{-1}(p)$.

If the number of clients n goes to infinity, σ converges monotonously decreasingly to

$$\sigma \xrightarrow[n\to\infty]{} \left(\frac{1}{m}\rho_D\, p(1-p)\right)^{1/2}$$

If, on the other hand, the number of periods m tends to infinity, σ apparently goes to zero. Thus, since the average default rates at single periods are independent and identically distributed by assumption, the overall mean default rate converges stochastically to the true probability of default by the law of large numbers. Note that it would be sufficient that average default rates are uncorrelated and have the same mean. Serial independence is not necessary for consistency. Also may dependencies among clients change over time as long as default probabilities remain constant.

However, at least the assumption of serially uncorrelated observations is crucial in the mean value model since otherwise a consistent estimation of default probabilities would be impossible. Together with the hypothesis that default

It is, therefore, desirable to have large rating categories, i.e. risk groups that contain many clients. It is particularly advantageous to be able to evaluate many periods independent from the size of the category.

Figure 4 shows the variance of the average default rate plotted against the number of clients per rating category for several numbers of periods. It appears that, independent of our example, 50 clients are sufficient to reduce the variance by as much as 98% of the maximum reduction implied in the number of clients.[112] 250 clients stand for 99.6% of the possible reduction.

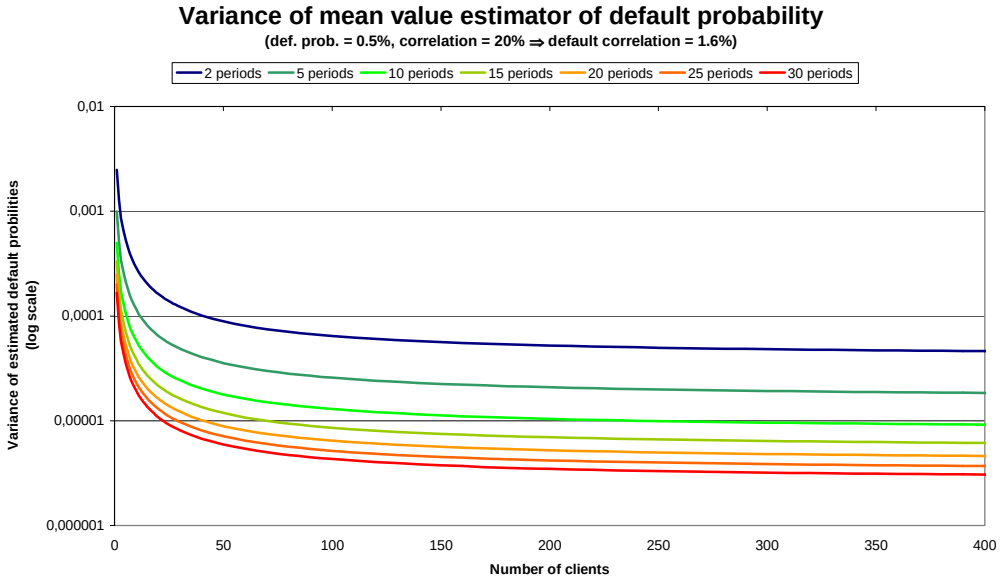

Figure 4: Variance of mean value estimator of default probability

If follows from the formulas in footnotes 112 and 113 that for a given number of counterparties per rating category the relative reduction of the estimator's variance is independent in the degree of dependence among counterparties while the absolute reduction[113] decreases. This observation is intuitively clear because, if clients strongly depend upon each other, their defaulting contains practically the same information meaning that an additional client offers very little new information while his share in the maximum possible variance reduction remains unchanged.

probabilities are constant over time, it is partially relaxed in models that relate a rating category's default probability to the state of the economy.

[112] This can be seen from the formula

$$\frac{\sigma^2(m,1,p,\rho_D)-\sigma^2(m,n,p,\rho_D)}{\sigma^2(m,1,p,\rho_D)-\sigma^2(m,\infty,p,\rho_D)}=1-\frac{1}{n}.$$

[113] The absolute reduction in the estimator's standard deviation is defined as

$$\sigma^2(m,1,p,\rho_D)-\sigma^2(m,n,p,\rho_D)=p\cdot(1-p)\cdot\left(1-\frac{1}{n}\right)(1-\rho_D).$$

Analogously, 5 periods of data diminish the estimator's variance by 80% of the maximum, 10 periods by 90% and 30 periods by 96.7%.[114] Note again that the impact of the number of periods of available data on the variance is unaffected by the degree of dependence among counterparties.

Table 1 also shows that for a given number of periods and clients in the analysis, the percentile that is represented by the true value of the probability of default increases with the level of dependence among counterparties.

In case of independence, the true value equals the median of the distribution of the estimator. Hence, one would expect to have a 50% chance to overestimate or to underestimate the correct value. If dependencies are rather elevated, though, it is considerably more likely to understate the accurate value than to overestimate it. Thus, especially in case of high correlations it is important to seek to diminish the estimator's variability.

d) How many rating categories should a financial institution distinguish?

Estimation errors implied by rating systems result from two major sources, the conceptual imprecision that rating categories are homogeneous and a sampling error from the actual estimation of default probabilities.

Although it is simplifying, the assumption of homogeneous categories is necessary in the mean value model to consistently perform all subsequent estimations. In order to use rating systems efficiently in this context, one is, therefore, forced to try to simultaneously keep both errors small.

As we have seen from the discussion, the supposition that all counterparties within one rating category are equal as far as default probabilities are concerned is less problematical the more categories there are. With an increasing number of categories the mean default probability converges to the individual default probability of each counterparty. From this point of view, many rating categories are good.

On the other hand, we have seen that the sampling error from the estimation of default probabilities resulting from the random nature of observed default frequencies decreases with the

[114] The relative reduction of the standard deviation in the number of available periods of data for a given number of clients in the portfolio is equal to

$$\frac{\sigma^2(1, n, p, \rho_D) - \sigma^2(m, n, p, \rho_D)}{\sigma^2(1, n, p, \rho_D) - \sigma^2(\infty, n, p, \rho_D)} = 1 - \frac{1}{m}.$$

number of clients per group. Hence, from this point of view, many clients per category are good. This implies, however, that the total number of clients in the portfolio should not be divided by too large a number of categories.

Both objectives are, thus, completely contradictory. Nevertheless, to optimize results a number of rules can be set up:

1. there should be at least 50-100 clients per rating category. If there are fewer clients, the variance of the estimator of default probability increases drastically due to the small number. This requirement limits the number of categories.
2. use rating categories efficiently. An 'investment' grade should be defined for areas of high exposure concentration. Distinguish more categories within the investment grade than in the remaining speculative grade. This reduces the estimation error on the portfolio level.
3. if a bank is too small to provide the necessary number of clients, another bank could be found with a similar structure so that they can pool their data to perform the estimations.
4. to facilitate a closer cooperation between financial institutions, it should be possible to render the first step in the rating process, i.e. the evaluation of clients' risk profiles and the assignment of a risk score, transparent to other banks.
5. not controversial, but also crucial is an effective early warning system to be sure that actual rating assignments are up to date and exposures are correctly calculated.
6. finally, the rating process should be stable over time to allow the maximum number of periods of default experience to be included in the analysis.

The number of rating categories that a financial institution should distinguish depends predominantly on its size and on the structure of its business. Large banks can use more categories than small banks. Institutions that hand out considerable amounts of money to all risk grades need more categories than banks with a rather conservative profile.

5. Rating based estimation of default probabilities: Credit Risk +

An important assumption in the mean value model was that default probabilities of each rating grade are constant over time, only observed default frequencies vary from period to pe-

riod. A different approach was chosen in the Credit Risk + model developed by Credit Suisse Financial Products[115] between 1993 and 1996.

a) Main concept

In this model, default probabilities are themselves stochastic. Each period nature is thought to independently draw a default probability from a probability distribution. Only as a second step actual defaults occur depending on the sampled default probability. Because default behavior is found to differ significantly between industries, the law from which a default probability is drawn depends on rating category and sector.

The variation of default probabilities is explained by background factors upon which clients systematically depend[116]. Their nature is not further specified and remains anonymous. All necessary information concerning systematic influences on default behavior is assumed to be contained in the probability law determining the development of default probabilities.

Note that although default probabilities are random in the model, the probability law that controls them is presumed to be constant. It seems paradoxical, but it is essentially the concept of continuity that is different in Credit Risk + from the mean value model.

b) Derivation of default probability

There are two main driving forces behind the conceptual architecture of Credit Risk +: firstly, the *analytical* derivation of the loss distribution for a portfolio of clients and, secondly, the mathematical necessity when it comes to estimating the required inputs. Hence, technical reasons have a very high priority in Credit Risk +.

The model supposes default probabilities to follow a gamma law[117]. The motive for this assumption is the well known compatibility of the gamma distribution to the Poisson distribution that leads to the desired analytical result for the portfolio loss distribution. The Poisson law is relevant in the second step of the modeling process. It describes the number of defaults per period in an industry/rating segment in Credit Risk +[118] once the probability of default is fixed.

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115 See Credit Suisse Financial Products (1996). Credit Risk + is mainly an adaptation of the compound Poisson model frequently used in insurance mathematics.

116 See Credit Suisse Financial Products (1996), p. 20.

117 This choice might surprise at first sight because the gamma law takes values on the whole positive real axis with positive probability. The probability to draw a default probability that is greater than 1 is negligible in realistic situations, though.

118 For technical details confer to Credit Suisse Financial Products (1996), p. 32ff.

The supposition of the Poisson law as the distribution of the number of defaults for a known probability of default is only justified if defaults are independent. Indeed, if default probabilities are fixed and equal for all clients[119], and if counterparties are independent, it can be shown that the number of defaults in that period follows a Poisson distribution[120]. If, however, counterparties are not independent, this implication would be wrong.

The assumption of independent defaults conditional to a given default probability is crucial in Credit Risk + also for a further cause. The gamma law from which default probabilities are drawn has two free parameters, both of which are functions of the expected value and the standard deviation of default probability.[121] The expected value can be consistently estimated by the mean of default frequencies as long as they are serially uncorrelated no matter how defaults depend upon each other at a certain point in time[122]. For the standard deviation things are more difficult. It is a problem to derive an estimator for it if the degree and the exact form of dependence between defaults are not fully specified. In case of independence, however, all joint distributions of defaults are known.

c) Discussion

In Credit Risk+, the focus is not on the exact modeling and estimation of default probabilities. The authors do not say much about the estimation of hidden input variables, especially of the volatility of default probabilities[123]. This is so much more fundamental as this task is not selfexplicatory despite the strict assumptions.

In his comparison of Credit Metrics and Credit Risk + Michael Gordy describes a way to estimate the standard deviation of default probabilities.[124] The estimator is given as

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119 This is again the assumption of homogeneity in a risk group that already occurred in the mean value model. Note that in this context a risk group is defined as a combination of rating and sector.

120 The Poisson approximation of the binomial distribution is an asymptotic result for the number of clients going to infinity. Hence, it would also be required that the number of clients in a segment is arbitrarily large. This turns out to be a problem in the long run because in Credit Risk + segments are further subdivided by exposure size in order to be able to calculate their loss distributions merely from knowing the distribution of the number of defaults in the segment. In consequence, the loss distribution resulting from the portfolio analysis in Credit Risk + can, firstly, take values with positive probability that are larger than the portfolio value and, secondly, the loss distribution is less discontinuous than it would be implied by real exposures because the jumps are filled up with imaginated defaults of small exposures. For further discussion see below section II.A.3.a).

For the estimation of default probabilities and volatilities, exposure size in not relevant, however, so that this problem can be neglected here.

121 See Credit Suisse Financial Products (1996), p. 45.

122 This follows from the law of large numbers.

123 See Credit Suisse Financial Products (1996), p. 12f. Apparently, the authors suppose that practitioners would use results published by rating agencies. For not publicly rated companies or private customers this might be doubtful.

124 Gordy (2000), p.24.

$$\hat{V}(p) = \frac{\hat{V}(\hat{p}) - \eta \bar{p}(1-\bar{p})}{1-\eta},$$

where $\hat{V}(p)$ stands for the estimated variance of *p*, *p* for default probability, $\hat{V}(\hat{p})$ for the estimated variance of the observed default frequency in period *t*, $\bar{p}$ for the average observed default frequency, and η for the average number of counterparties in the industry / rating segment[125].

Note that this variance estimator leads to invalid results if

$$\hat{V}(\hat{p}) < \eta \bar{p}(1-\bar{p})$$

where the estimated variance takes on negative values. Simulation results show that this happens more frequently the smaller default frequencies and true default volatilities are and the fewer periods are available for evaluation.[126]

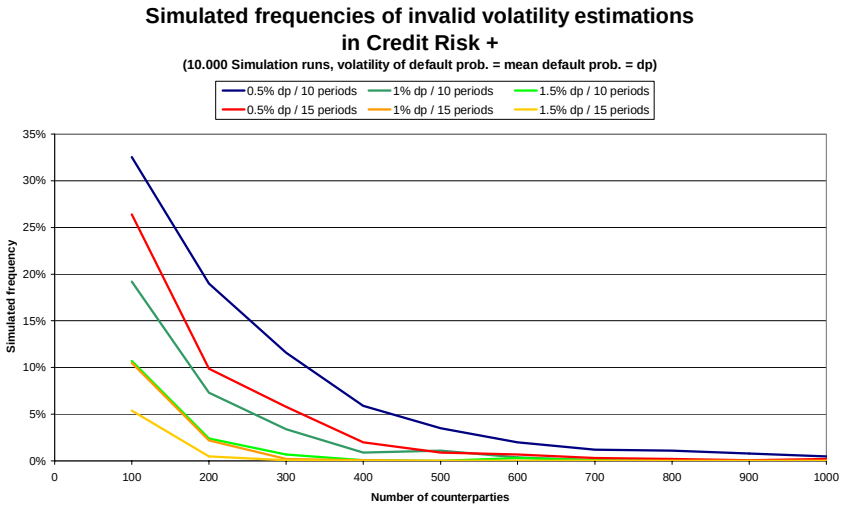

Figure 5: Frequency of invalid volatility estimations in Credit Risk +

[125] Let n_i be the number of firms in the industry / rating-segment in period $i = 1,\ldots, m$. Then η is given as

$$\eta = \frac{1}{m} \sum_{i=1}^{m} \frac{1}{n_i}$$

[126] Default probabilities were simulated from the gamma law with default volatilities being equal to default probabilities. In this case the density function of the gamma distribution has the simple form

$$f(x) = \frac{1}{\sigma} e^{-\frac{x}{\sigma}}$$

with σ being the volatility of default probabilities. Random variates can be easily generated by inversion of the cumulative distribution function. Having drawn a default probability for one period, independent defaults were simulated for the respective number of counterparties. This was repeated for the required number of periods and the results served as input for the estimation of default volatility. The whole procedure was repeated 10.000 times to generate the distribution of the estimator.

It also turned out that the volatility estimator in Credit Risk + is consistent, but biased in small samples. The volatility is expected to be largely understated if the number of clients and the number of periods are small. Having 5 or more periods of data, the estimation error caused by a small number of clients seems to be minimized if there are at least 200 clients in the industry / rating segment. Due to the decomposition of the portfolio by two criteria, rating and industry, this implies a financial institution that distinguishes 10 rating grades and has customers in 15 sectors to have at least 30.000 clients evenly distributed over the risk groups. This is a harsh requirement since there are always some segments that are only sparsely populated. In these risk groups a significant estimation error is to be expected.

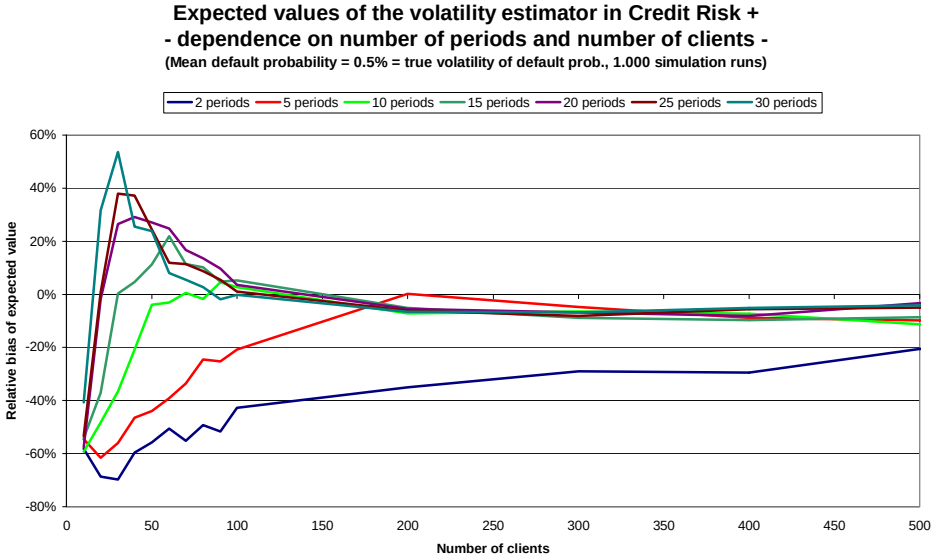

Figure 6: Bias of estimated default rate volatility in Credit Risk + dependent on number of periods and number of clients

It is remarkable that for 30 to 50 clients in a segment the default rate volatility is overstated if there are many periods of data. This is because the components of the estimator that increase or decrease, resp., in the number of periods and / or clients superimpose so that the estimated volatility of default reaches a peak if the segment is middle-sized and if the time series is long.

Figure 7 illustrates that the bias in the estimation of default volatility caused by a small number of periods of data is slow to disappear. Having observations over 15 periods, the volatility is still expected to be systematically understated by as much as 5%. Taking into consideration that default volatility is one of the main risk drivers in Credit Risk +, this result is quite worrying because only very few banks have this long history of observations.

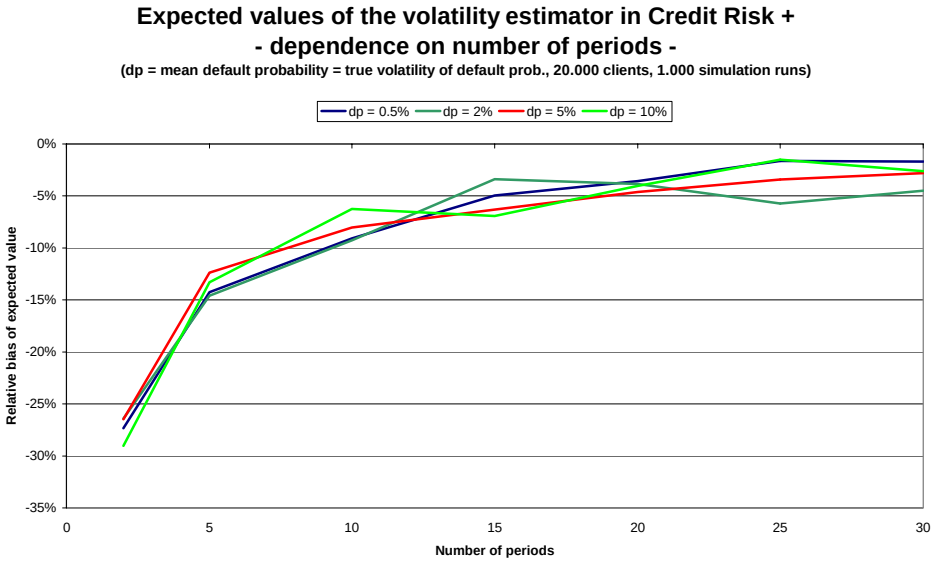

Figure 7: Bias of estimated default rate volatility in Credit Risk + dependent on number of periods

Finally, although default probabilities in a certain period are supposed to depend upon certain systematic risk factors, it is assumed in Credit Risk + that default probabilities are serially independent. From an economist's points of view, these postulations seem to be inconsistent and contradictory because they imply that also those "systematic" factors are independent from period to period. This is intuitively not very compelling, albeit the fact that they remain anonymous. The assumption of serial independence is, however, crucial to the model. Otherwise the random law controlling default probabilities in a certain period would be conditional to the actual values of the systematic factors, which in turn would require precise information about these factors and how to measure them.

6. Rating based estimation of default probabilities: Credit Portfolio View

Another approach to model dependence of default probabilities on systematic risk factors was chosen by Credit Portfolio View developed by Thomas Wilson[127] and published by McKinsey & Company in 1997.

a) Main concept

Most importantly, Credit Portfolio View tries to model an *empirical* relationship between firm's default behavior and the macroeconomic indicators of the business cycle.

Indeed, business cycle effects can sometimes be clearly seen in default data.

[127] Confer to Wilson (1997a) and Wilson (1997b).

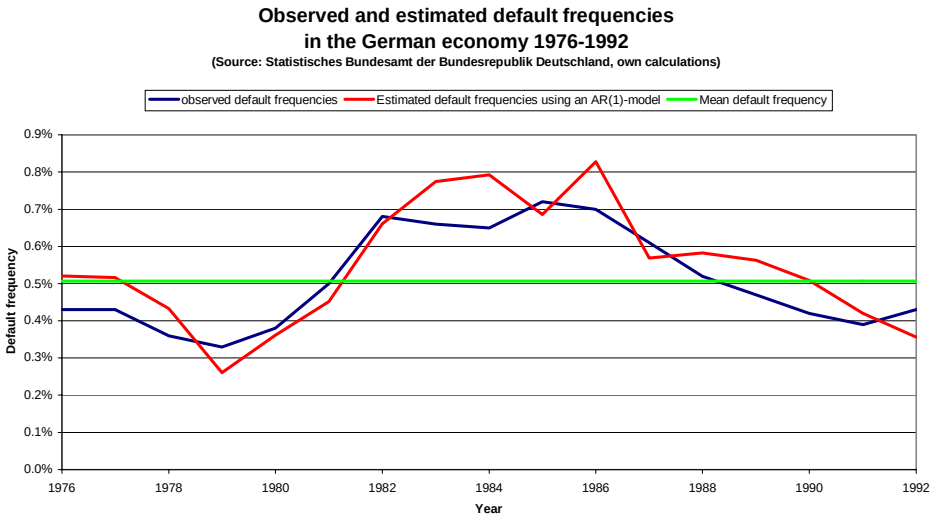

Figure 8: Observed and estimated default frequencies in the German economy 1976-1992

Figure 8 shows annual default rates for the entire German economy from 1976 to 1992[128]. From 1976 to 1981 default rates are well below average, from 1981 to 1988 they are considerably higher than average and in the period from 1988 to 1992 they sink once again. Therefore, using the mean value model, default rates are systematically overestimated in good periods and also heavily understated during recessions. The idea to relate observed default frequencies to macroeconomic data is, thus, well founded in the data.

However, two important adaptations have to be made. Producing different kinds of goods and services, all industries are not equally integrated into the macroeconomy. Wilson shows that there are major differences in annual fluctuations of default rates across industries finding that, for instance, construction is strongly and energy and mining are hardly affected by the business cycle[129]. The analysis of companies default behavior should, therefore, take account of their industry.[130]

It is also a well known fact that business cycle effects strongly depend upon a firm's general financial condition. Firms with a weak market position are hit much harder by a recession than highly competitive companies. Indeed, a default of an actual AAA-company has never been observed no matter what the macroeconomic environment was. What is more, the macroeconomic dependence of investment grade companies is hardly detectable. Hence, Wilson chooses to use an industry's speculative grade companies as an indicator of that industry's

[128] The data comprises all VAT payers in Germany, i.e. altogether around 4 mio. firms of all sizes, ratings, and industries. For details confer to Bär (2000), p. 2.

[129] Wilson (1997a), p.112f.

[130] Wilson does not discuss the case of companies that operate in several industries.

economic health. In a further step he then indirectly derives default and migration probabilities of specific rating grades in that industry.

However, Credit Portfolio View goes further. The aim is not only to explain firms' actual default behavior, but also to forecast their default probabilities over the whole life of the longest lasting contract in the bank's portfolio, this is well over 10-30 years in the future[131]. It is argued that some contracts are highly illiquid so that a bank is forced to hold them to maturity, even if the credit quality of the counterparty deteriorates. In order to capture this kind of contract's credit risk appropriately, the client's default probability has to be adapted to its time horizon.

b) Derivation of default probability

In order to achieve this, Credit Portfolio View proceeds in three steps[132]:

1. It begins with a model of the future development of the relevant macroeconomic factors[133]. The model is fitted using historical macroeconomic data. By drawing prediction errors randomly, the macroeconomic development is simulated over the desired time horizon.
2. Secondly, a multi-factor model is chosen to describe the relationship between annual default rates of speculative grade companies in each industry and macro factors. Again, the model is fitted using historical data, and future annual default probabilities of speculative grade companies are simulated by random choice of prediction errors.
3. Finally, the simulation results are used to calculate conditional default and migration probabilities for each year and rating grade up to the end of the time horizon.

For each country / industry combination three macroeconomic factors are chosen und modeled by univariate autoregressive processes such as AR(p) or ARIMA(p,d,q). All parameters are estimated with least squares techniques. The vector $\varepsilon_{macro,t}$ of estimation errors of the fitted factor models is assumed to be normally distributed with mean 0 and covariance matrix Σ_{macro}. Σ_{macro} has to be estimated from historical data. To extend the factors' historical time series into the future, error terms $\varepsilon_{macro,t}$ are simulated by independent draws from the specified distribution using standard techniques.

131 Wilson (1997a), p. 113.

132 Wilson (1997a), p. 114

133 This model may include global factors, but it is designed to explain the country (and industry) specific default behavior.

To make sure that all simulated future default probabilities take values between 0 and 1, observed speculative default rates are mapped to the whole real axis by the logit transformation

$$y_{t,i} = L(p_{t,i}) = \ln\left(\frac{1-p_{t,i}}{p_{t,i}}\right)$$

where $p_{t,i}$ is the observed default rate in period t and sector i and $y_{t,i}$ is the transformed default rate[134]. $y_{t,i}$ is then linearly regressed against the explaining macroeconomic variables $X_{1,i,t}$,…, $X_{n,i,t}$ relevant for sector i

$$y_{t,i} = \beta_{o,i} + \beta_{1,i} X_{1,i,t} + \ldots + \beta_{n,i} X_{n,i,t} + \varepsilon_{def,i,t}$$

where $\beta_{0,i}$, ..., $\beta_{n,i}$ are unknown parameters. Again, the parameters are estimated with least squares techniques applied to the transformed default rates[135], and error terms are assumed to be normally distributed with mean 0 and covariance matrix Σ_{def}. Future values of $y_{t,i}$ are simulated by drawing independent error terms from this distribution[136] and then retransformed into default probabilities by the inverse logit function

$$p_{t,i} = L^{-1}(y_{t,i}) = \frac{1}{1+\exp(y_{t,i})}.$$

The last step, the derivation of conditional transition matrices is not well documented in Wilson's publications. It is merely indicated that the ratio of the simulated speculative default probability $p_{t,i}$ for the future period t to the mean default probability $\overline{p_i}$, thus,

$$\frac{p_{t,i}}{\overline{p_i}},$$

serves as an indicator as to what extent transition probabilities might have changed compared to the long term mean due to the simulated macroeconomic background, from which a conditional annual migration matrix $M\left(\frac{p_{j,i}}{\overline{p_i}}\right)$ is obtained. A Markov assumption[137] then yields a t-year cumulative transition matrix

[134] Note that, as in Credit Risk +, defaults are supposed to be independent conditional to a macroeconomic scenario.

[135] Cf. to Bär (2000), p. 2 and 12. This estimation technique is also used in McKinsey's software implementation of Credit Portfolio View, cf. McKinsey (1999), p. 36.

[136] Precisely, Wilson allows for covariances between ε_{macro} and ε_{def} and simulates all errors simultaneously.

[137] The Markov assumption means that a firm's transition probability in period t merely depends upon its actual rating at the beginning of period t and not on its full migration path in the past.

$$M_{t,i} = \prod_{j=1}^{t} M\left(\frac{p_{j,i}}{p_i}\right).$$

c) Discussion

The intention to relate firms' historic default frequencies to observable systematic risk factors such as macroeconomic indicators is clearly the main advantage of Credit Portfolio View in comparison to Credit Risk + or the mean value model.

For a number of reasons, it is, however, doubtful whether this goal really has been achieved.

(1) Modeling of macroeconomic processes

Credit Portfolio View describes the underlying macroeconomic factors by autoregressive processes, i.e. by processes whose future development is exclusively explained by their own history.[138] Aiming at a long-term prediction of factors and default behavior, this feature is clearly advantageous with regard to a software implementation for it renders the necessary simulations extremely easy to carry out as it does not require any further input. For two reasons, autoregressive processes are problematical choices as models for macro factors.

Firstly, from an economist's standpoint, it is uncertain whether the macroeconomic factors given as examples by Wilson[139] such as the unemployment rate, GDP growth rate, rate of government spending etc. can well be described relying only upon their own past. Even more than equity market data these factors strongly depend on factors such as the results of elections, on political decisions, and the will to carry out reforms. All of those events are contingent in nature and unrelated to the past. This is why it is so difficult to forecast them.

Secondly, the persistence of shocks and extreme values is relatively large in autoregressive processes. If they are used as a model for macroeconomic factors in a credit risk model, the effect of an economic crisis will be felt for many years.

Figure 9 compares the conditional mean default probability of speculative grade companies *t* years after a crisis with their unconditional mean default probability. In both cases it is assumed that reality functions exactly as Credit Portfolio View supposes and that all parameter values are known and do not have to be estimated[140]. It is clearly visible that 30 years after the

[138] And white noise.

[139] Wilson (1997a), p. 113.

[140] The macroeconomic factor is described by the AR(2)-process

crisis the economy still has not fully recovered and that the default probability remains well above average. It is an open question whether this behavior is a good description of long-term default rates.

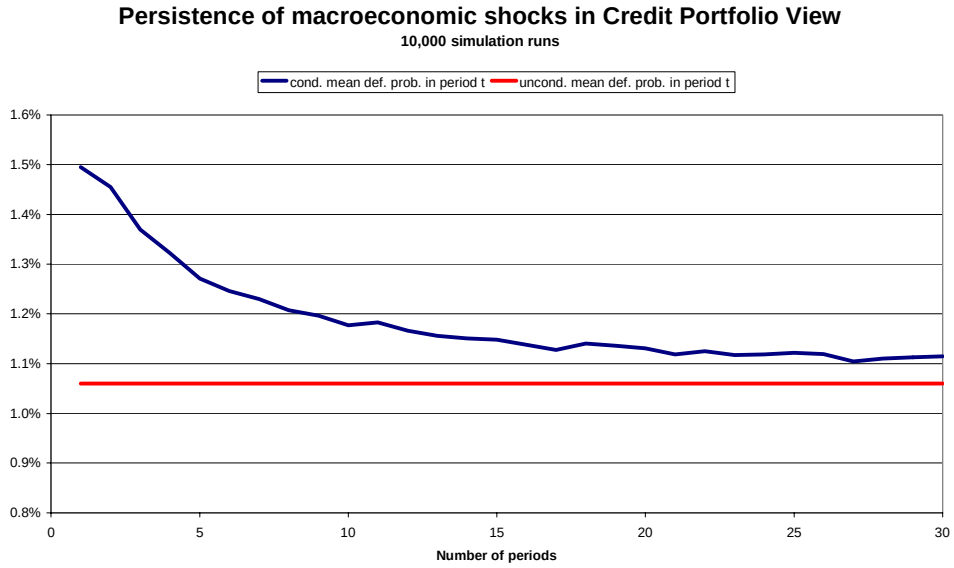

Figure 9: Persistence of macroeconomic shocks in Credit Portfolio View

(2) Relation of default rates to systematic factors

A key step in the regression model relating industries' speculative default rates to the macroeconomic background previously simulated is the logit transformation. The logit transformation has a very critical feature. It not only maps the unit interval to the real axis, but it is also non-linear, namely convex in [0,0.5] and concave in [0.5,1] (see Figure 10). It is important to note that probability distributions are not stable under non-linear transformations. In general, it cannot be seen from the functional form of the transformation how the moments and stochastic properties of the distributions are distorted.

$$X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_X = 0.2 X_{t-1} + 0.2 X_{t-2} + \varepsilon_X$$

where ε_X is normally distributed with mean zero and standard deviation 0.9. The logit transformation of the default probability is assumed as

$$Y_t = \beta_0 + \beta_1 X_{t-1} + \varepsilon_Y = 5 + X_{t-1} + \varepsilon_Y$$

where ε_Y is again normally distributed with mean zero and standard deviation 0.3. The crisis is described by the starting values $X_0 = X_{-1} = -1$.

Note that the persistence of crises is increasing in α_1, α_2, and β_1 and decreasing in the standard deviations of the error terms.

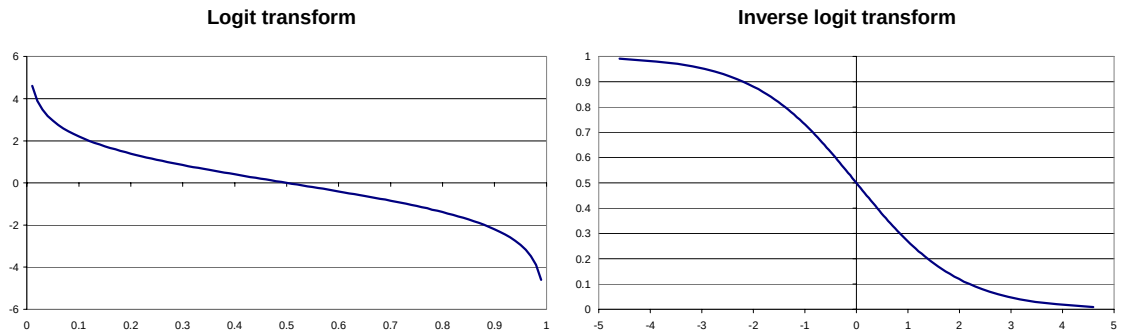

Figure 10: Logit and inverse logit transformation

Realistic default probabilities can be assumed to be smaller than 50% even in bad times, thus, the logit transformation is convex in the relevant area. To see the effect of a convex transform on a distribution, assume an arbitrary random variable X with expectation μ. Let $g(x)$ be a convex transform defined in the domain of variation of X. Then there exists a line through the point $(\mu,g(\mu))$ that lies fully below the graph of g (see Figure 11).

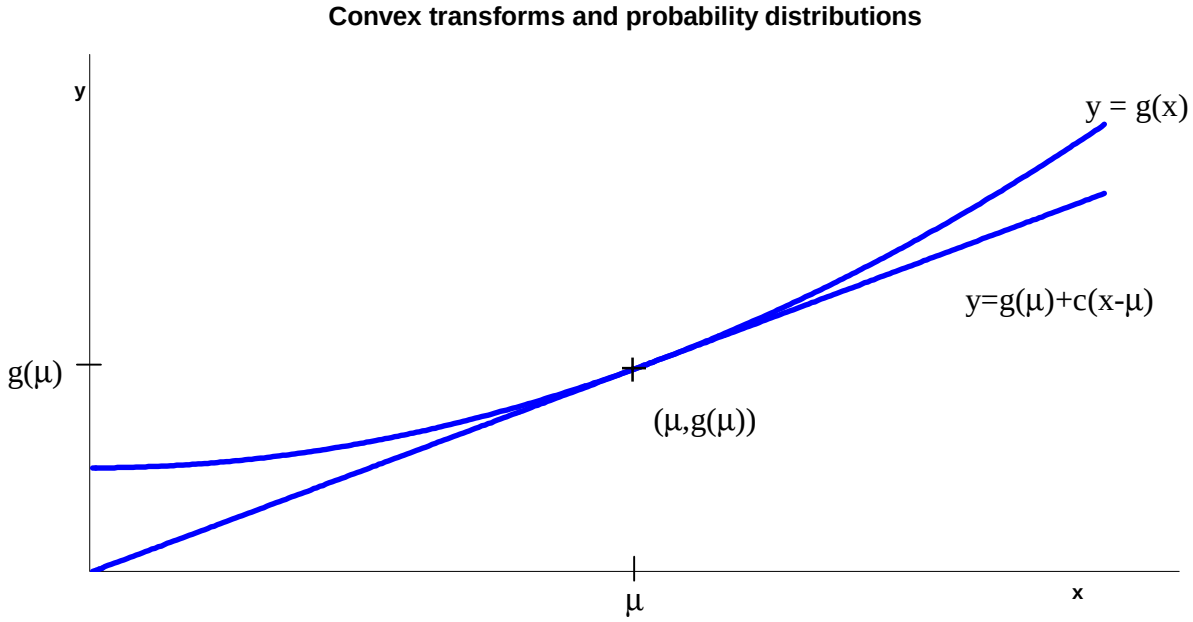

Figure 11: Convex transforms and probability distributions

Thus,

$$g(x) \geq g(\mu) + c(x - \mu)$$

for all x. If $Y = g(X)$ has finite expectation, it follows from the monotonicity and linearity of the Lebesgue-integral that[141]

[141] This result is known as Jensen's inequality.

$$E\big(g(x)\big) = \int_{-\infty}^{\infty} g(x)dF(x) \geq \int_{-\infty}^{\infty} [g(\mu) + c(x-\mu)dF(x)]$$

$$= g(\mu) \underbrace{\int_{-\infty}^{\infty} dF(x)}_{=1} + c \underbrace{\int_{-\infty}^{\infty} (x-\mu)dF(x)}_{=0} = g(\mu)$$

Hence, we have

$$E\big(g(x)\big) \geq g(\mu).$$

Note that there is equality only for deterministic random variables, i.e. if $P(X=\mu) = 1$.

This implies for Credit Portfolio View that the expectations of each period's transformed default rates conditional to the actual macroeconomic setting are greater than the transformed default probabilities. This happens unless the exact conditional default probabilities are observed each period with probability one. Although conditional independence had been assumed[142], it follows from the law of large numbers and the central limit theorem that this is only the case if there are infinitely many speculative grade companies in each industry in the financial institutions portfolio.

If there is only a finite number of speculative grade companies in each industry, default rates cannot be understood as default probabilities. The regression against the macroeconomic factors is then performed with expectations of too high transformation results of observed default rates. Since a linear regression is unbiased by construction, the estimated regression line will be above the true regression line, even if the model is correctly specified[143]. Hence, the values that are simulated according to the estimated regression line are too, in turn, too high.

Finally note that the inverse logit transformation decreases strictly monotonously. If the simulated values are too high then the retransformed values, i.e. the simulated default probabilities, are too low due to the decreasing retransformation.

Thus, if there are only a finite number of speculative grade companies in each industry, estimation results, and in consequence simulation results, are biased towards too low probabilities of default. The default risk is systematically underestimated in Credit Portfolio View.

[142] If the model is not exact in that respect that macroeconomic factors don't fully explain dependencies between clients, the assumption that one period's default rates are a consistent estimator of that period's default probability is generally invalid. Confer to the discussion in footnote 111.

[143] I.e. the error occurs even if reality functions exactly as the model supposes.

(3) Example

To illustrate this effect, we suppose that the true relationship between the macroeconomy and firms' default behavior is exactly as described in Credit Portfolio View. This means, we do not take account of what was said at the beginning of the discussion and assume that there is no general modeling error.

We consider an industry whose default behavior depends only on one macroeconomic factor. Let's presume that this factor *X* follows an AR(2)-process, precisely let

$$X_t = 0.4X_{t-1} + 0.4X_{t-2} + 0.1\varepsilon_X$$

where ε_X is i.i.d. standard normally distributed. In order to isolate the systematic estimation error in the next step, we assume that the parameters of the macroeconomic process are known so that no further error at this point occurs.

The true regression model that relates transformed default probabilities to the macroeconomic factor is set as

$$Y_t = 5 + X_{t-1} + \varepsilon_Y .$$

The error term ε_Y is supposed to be i.i.d. normally distributed with mean zero and known standard deviation $\sigma = 0.3$. Note that only the last period's realization of the systematic factor is needed to estimate or predict the present period's transformed default probability. Again we assume that the general form of the regression model in known. Only the two parameters $\beta_0 = 5$ and $\beta_1 = 1$ are to be estimated.

While the long-term mean probability of default depends on both parameters, β_0 certainly has the dominant influence. Since the inverse logit transformation decreases monotonously, high values of β_0 correspond to low default probabilities.[144]

In order to fit the model, a historical time series of *n* periods of unknown default probabilities and observed default frequencies for various numbers of counterparties in the industry was simulated upon which the estimations could be based. Having performed the estimations, default probabilities were simulated for the next *n* periods. The additional problem of a prediction horizon much longer than the history of observations was avoided.

[144] The long-term mean probability of default is increasing in β_1.

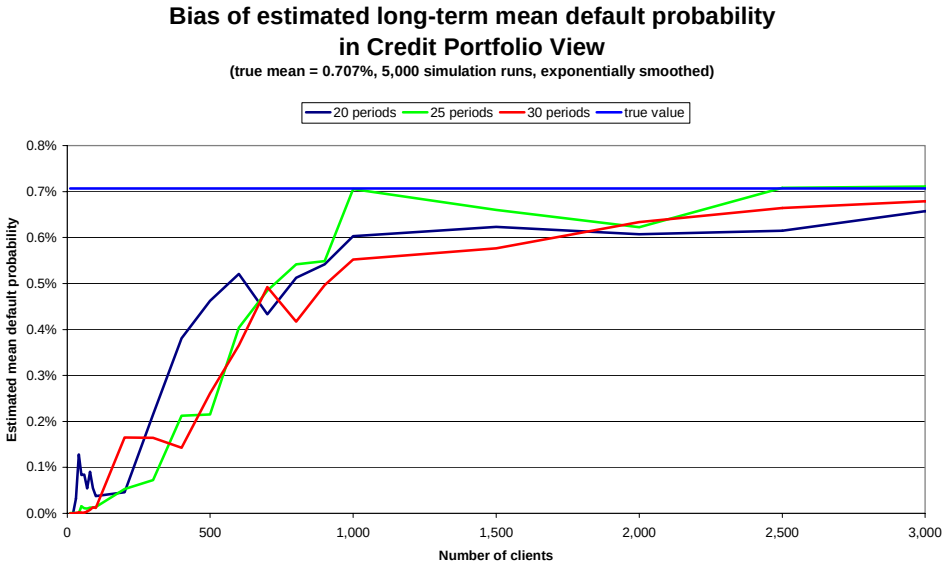

Figure 12: Bias of long-term mean default probability in Credit Portfolio View

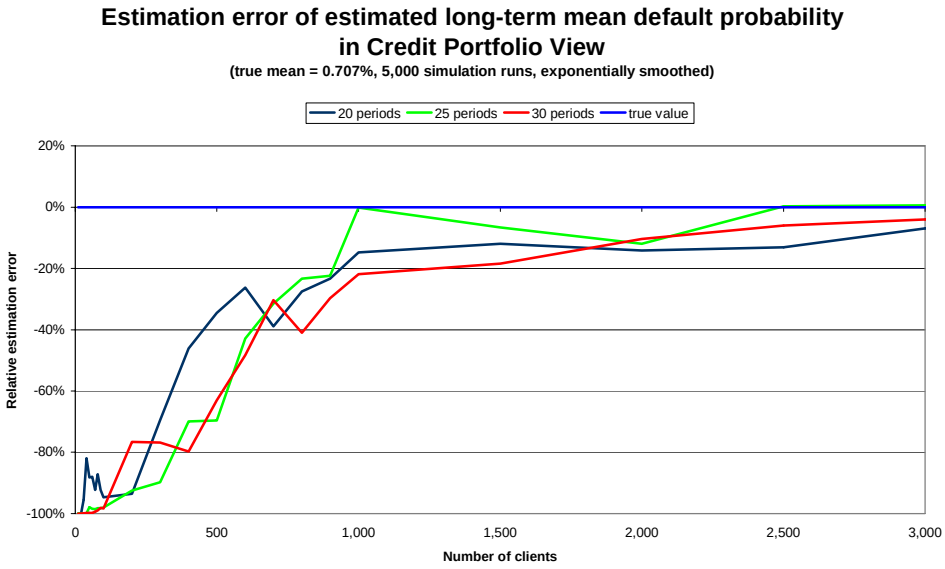

Figure 13: Relative error of estimated long-term mean default probability in Credit Portfolio View

Figure 12 and Figure 13 clearly show that the long-term mean probability of default of the process is consistently understated, even if there are up to 3,000 speculative[145] grade firms in a particular industry in the bank's portfolio. This amount is certainly very rare. Especially the relative estimation error is considerable since the quantity to be estimated is so small.

The picture is even more transparent if we look at the parameter estimates[146].

[145] Moody's Investors Service and Standard and Poor's define a speculative grade company as having a lower rating grade than Baa or BBB, respectively. This corresponds to a default probability of more than 0.2-0.3%.

[146] Note that the bias of parameter estimates does not depend on the length of the prediction period.

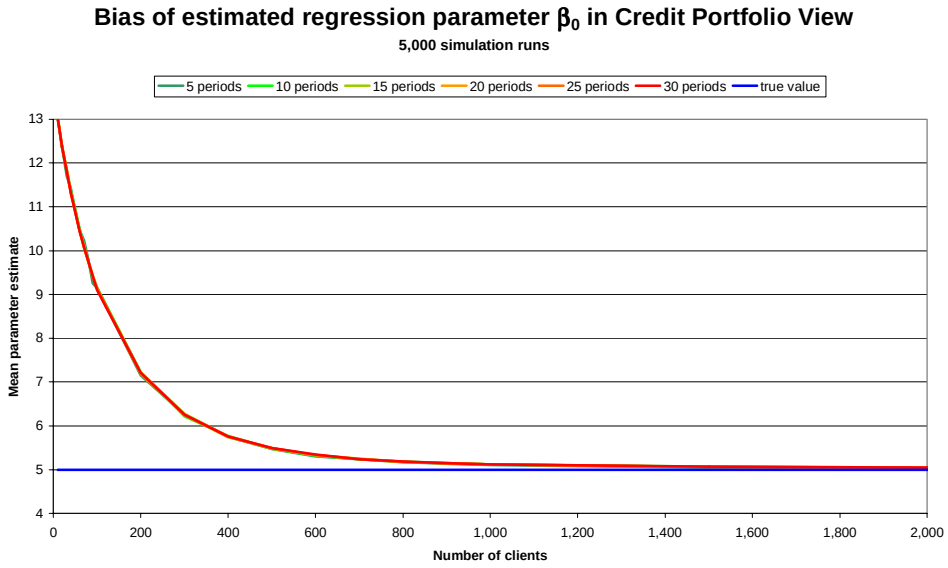

Figure 14: Bias of estimated regression parameter β_0 in Credit Portfolio View

As it should be, the expected value of the estimate of β_0 does not depend on the length of the time series of historical observations. This comes from the fact that the AR(2)-process describing the development of the systematic risk factor is stationary and, therefore, the true default probabilities and the observed default frequencies are too. Since the estimation of the regression parameters of the transformed default rates is unbiased the deviation of the expected value of the parameter estimate of the true parameter value, i.e. the bias, is merely owed to the non-linearity of the logit transformation. As already mentioned, large values of β_0 correspond to low default probabilities.

Albeit the mean of the parameter estimate does not depend upon the number of periods of data available for analysis, its standard deviation does.

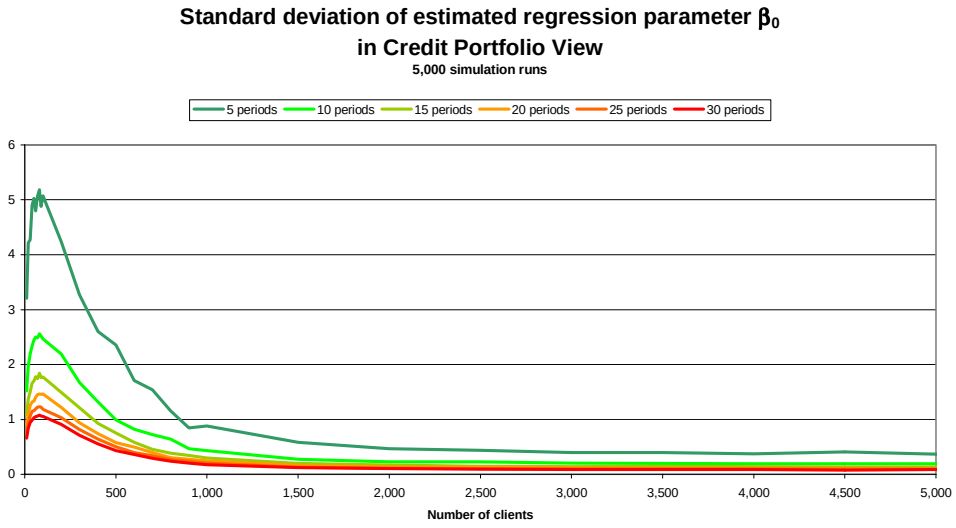

Figure 15: Standard deviation of estimated regression parameter β_0 in Credit Portfolio View

Having few periods of data and few clients the variation of the estimate is remarkable. With more than 3,000 clients in the segment, however, its variability almost exclusively depends on the number of periods in the sample.

If the number of clients is very small, the probability of not observing a default at all is relatively high even if each individual company's default probability is quite high[147]. This is why the volatility of the estimate declines again for a small number of counterparties.

The result is similar if we look at β_1.

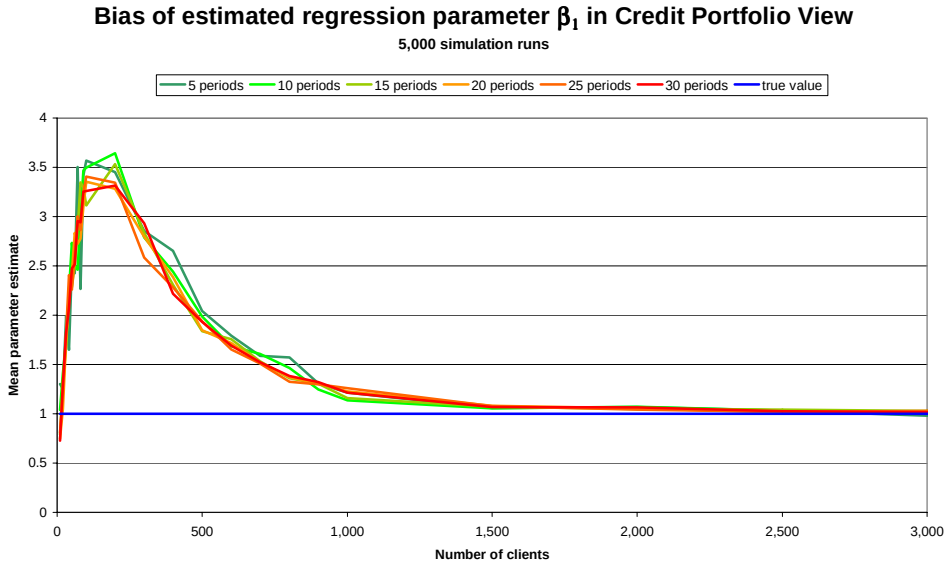

Figure 16: Bias of estimated regression parameter β_1 in Credit Portfolio View

[147] If there are 50 independent counterparties, each defaulting with 2% probability, the likelihood of observing no default is 36.4%.

Here again the mean estimation error does not depend on the number of periods of historical data. It is noteworthy, though, that far more clients are necessary to reduce the bias than in case of β_0.

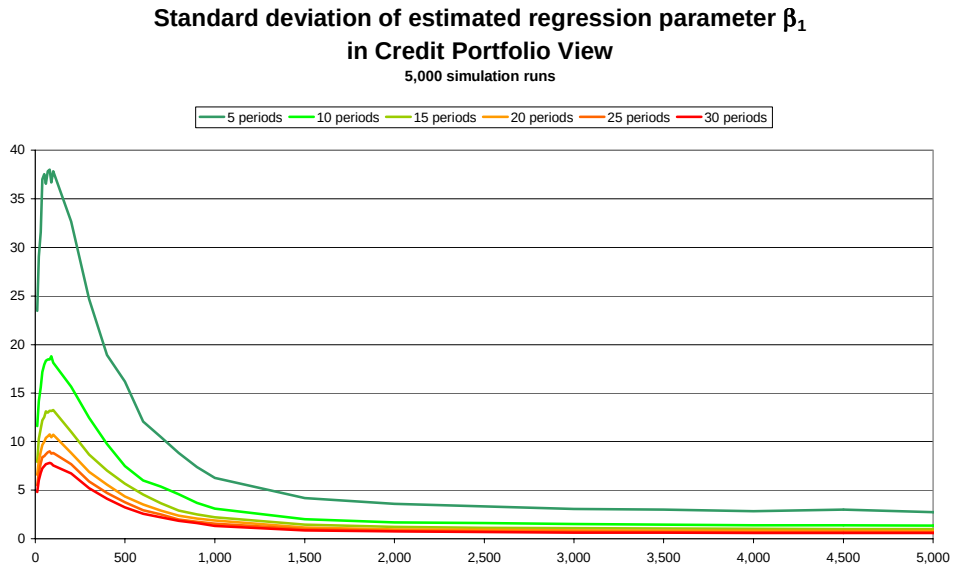

Figure 17: Standard deviation of estimated regression parameter β_1 in Credit Portfolio View

The volatility curve of the estimate of β_1 has a similar shape as the one of β_0. The volatility compared to the size of the parameter is much higher. This comes from the error term ε_Y being a second source of variability in the regression besides the systematic factor. Its non-systematic influence can only be neutralized by a long history of observations.

The distortion of the estimates is also visible if we consider both parameters simultaneously.

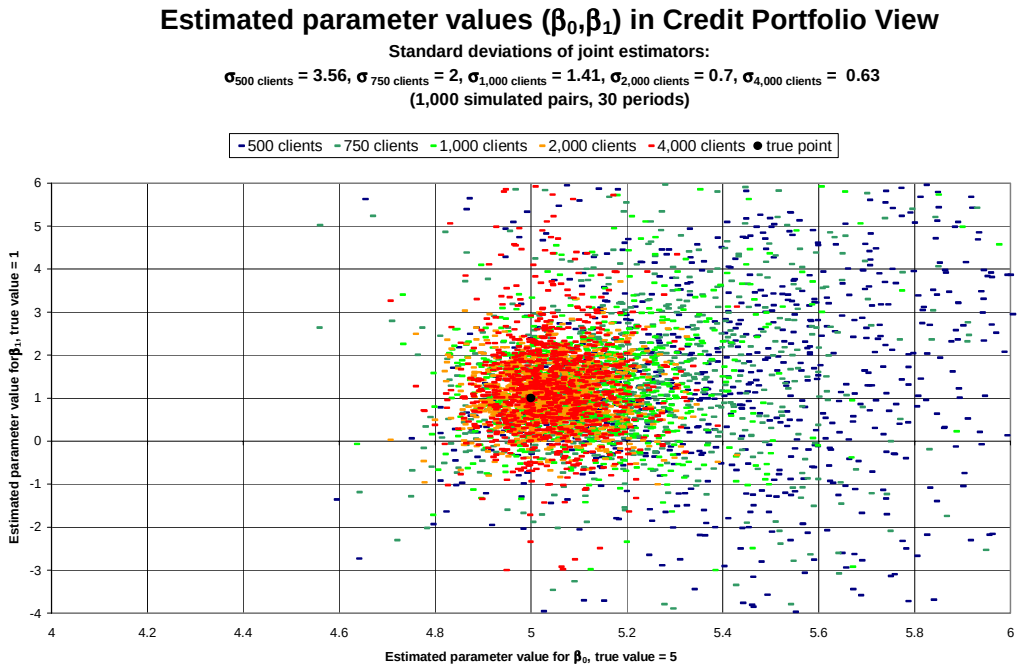

Figure 18: Distributions of estimated parameter values for (β_0,β_1) in Credit Portfolio View

Especially for small numbers of clients it can be clearly seen that the distributions assign more probability mass to large values of (β_0,β_1). For β_0 this effect is even more pointed than for β_1.

The fact that the bias of the parameter estimates and the influence of the number of clients on the estimates' standard deviations disappear if there are several thousand counterparties, results from the almost sure convergence of a period's observed default rate to its default probability when the number of clients tends to infinity. It is remarkable that the logit transformation is so sensitive to deviations of default rates from default probabilities that so many counterparties are needed to make the systematic estimation error vanish.

To obtain another example of the distorting effect of the logit transformation on the pair of parameters (β_0,β_1), we can consider the distributions of the long-term mean default probabilities that are simulated after the parameters were estimated.

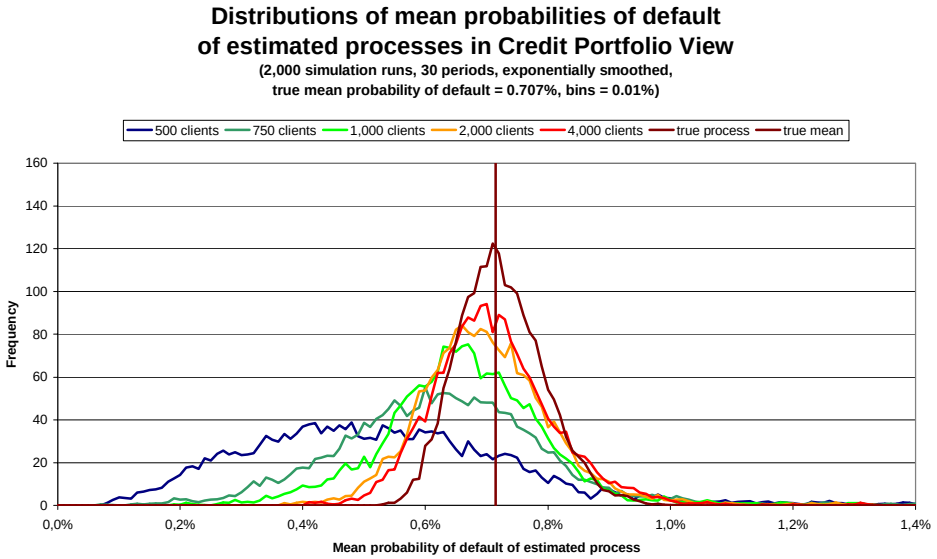

Figure 19: Distributions of long-term mean probabilities of default of estimated processes in Credit Portfolio View

The plots displayed in Figure 19 were produced as follows. The true process was constructed as in the example above. Parameters (β_0,β_1) were estimated based on a 30-year history of observed data using a sample of 500 to 4,000 speculative grade companies in the industry under consideration. Then default probabilities were simulated over the next 30 years and their mean was calculated. This procedure was repeated 2,000 times to yield the distributions of the long-term mean default probabilities.

	500 clients	750 clients	1,000 clients	2,000 clients	4,000 clients	true distribution
Expectation	0,58%	0,64%	0,65%	0,69%	0,70%	0,71%
Standard deviation	1,07E-02	4,29E-03	1,69E-03	1,10E-03	9,97E-04	6,83E-04
Skewness	3,89E-05	1,95E-06	2,44E-08	5,31E-10	3,98E-10	1,19E-10
Kurtosis	1,57E-05	2,94E-07	8,09E-10	9,80E-12	5,21E-12	6,69E-13
Range 0	0,07%	0,07%	0,18%	0,37%	0,40%	0,53%
5%- percentile	0,20%	0,34%	0,43%	0,53%	0,55%	0,60%
20%- percentile	0,33%	0,48%	0,55%	0,60%	0,62%	0,65%
40%- percentile	0,44%	0,57%	0,62%	0,65%	0,67%	0,69%
50%- percentile	0,50%	0,61%	0,65%	0,68%	0,69%	0,70%
60%- percentile	0,56%	0,65%	0,67%	0,70%	0,71%	0,72%
80%- percentile	0,70%	0,74%	0,75%	0,76%	0,77%	0,76%
95%- percentile	0,98%	0,90%	0,86%	0,87%	0,87%	0,83%
Range 1	42,66%	16,23%	4,15%	1,45%	1,31%	0,97%

Table 2: Characteristics of distribution of long-term mean default rates in Credit Portfolio View

The characteristics of the distributions are summed up in Table 2. Three things become apparent. Firstly, the expectation of the long-term mean default probabilities of the estimated processes is systematically understated. Secondly, the same observation holds true for the median of the distribution. The median of the true distribution corresponds to the 60%-

percentile if there are 4,000 clients in the sample up to the 80%-percentile if there are only 500. Thirdly, the bias decreases and standard deviation, skewness, kurtosis and span of the simulated distribution converge to the respective characteristics of the true distribution as the number of clients increases[148].

It's finally worth noting that the systematic estimation error in the regression of default rates against macroeconomic factors in Credit Portfolio View is not due to the precise functional form of the logit transformation, but rather to its non-linearity in combination with the estimation techniques used. Any other non-linear transformation would cause a similar effect. Hence, it would not help very much to replace the logit transformation by a probit transformation, for instance by the normal cumulative distribution function (cdf) which also only takes values in the unit interval[149]. Other than the logit function, the inverse normal cdf is concave in (0,0.5], so that the mean of the transformed default rates is smaller than their transformed mean[150]. The normal cdf itself, that retransforms the regression results, monotonously increases, so that the total estimation result understates the true values of default probabilities once again.

(4) Conditional transition matrices

Even if default probabilities were correctly predicted, this is not the final step. The estimation of default probabilities was not done for a specific rating grade, but for all 'speculative grade' companies together, i.e. for companies that are known to have different default and transition probabilities. Thus, individual rating grades need to be adapted to macroeconomic scenarios. For this purpose Wilson proposes another transformation of the estimated default probabilities of the speculative grade firms in the respective industry depending on the ratio of the simulated conditional default probabilitiy for a certain period and the estimated long-term mean default rate.

Having seen that transformations can have a significant impact on the properties of a distribution and the precision of a model, it is particularly critical that the derivation of conditional transition matrices is a gap in the documentation of Credit Portfolio View.

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148 Note that standard deviation, kurtosis and span are necessarily larger at the estimated distributions because of the unsystematic estimation uncertainty due to the finite number of periods in the sample that adds variability to the estimated distributions.

149 This is proposed in a context not related to Credit Portfolio View by Kim (1999). He suggests the normal cdf as a transform of default rates not to avoid estimation errors, but because Credit Metrics uses it to model rating migrations.

150 This is Jensen's inequality again.

Apart from another possible systematic estimation error, this transformation is the third successive estimation between macroeconomic risk factors and each specific rating grade's migration probability, especially its default probability, where each step increases the variability of the results. Let's take an

(5) Example

The future macroeconomic situation becomes more and more independent of the present one as the prediction horizon increases. Therefore, the simulated long-term mean cumulative probabilities of default of each rating grade should converge to the historically observed long-term averages. Using the Markov assumption mentioned above, *n*-year cumulative default probabilities can easily be derived from the annual transition matrices published by the rating agencies by multiplying the respective matrix *n* times with itself.[151] If we compare the outcome for a 10-year time horizon with the example given by Wilson[152] for the German economy, we see that the results differ greatly.

Rating[153]	Extrapolation of Moody's transition matrix[154]	Wilson's results	Ratio
AAA	0,47%	1,74%	374%
AA	1,35%	4,12%	305%
A	3,72%	9,58%	257%
BBB	9,71%	22,63%	233%
BB	24,78%	46,75%	189%
B	44,96%	69,12%	154%
CCC	68,55%	83,58%	122%

Table 3: 10-year cumulative default probabilities extrapolated using Markov assumptions

The deviation is so much more striking as Wilson himself uses a Markov assumption and Moody's annual mean transition matrix[155] to derive long-term transition matrices. Especially for investment grade companies, the probabilities derived from Moody's data only make up less than half of Wilson's findings. The transformation of speculative grade default probabilities in rating specific ones is apparently no straightforward exercise. It does not seem convincing to explain the results by a disastrous economic situation in Germany in 1997.

[151] Wilson himself quotes Moody's transition matrix, Wilson (1997a), p. 113, table A. We used this data for the extrapolation in Table 3. Note that Wilson quotes exactly the same matrix as Standard and Poor's transition matrix in Wilson 1997c, figure 6.

[152] Wilson (1997a), p. 117, table B.

[153] It is Wilson's terminology to use Standard and Poor's notation for Moody's rating grades (see also footnote 151).

[154] See also footnote 151!.

[155] It is actually not quite clear from Wilson's article which rating agencies transition matrix he uses for his example. Note, however, that Moody's, Standard and Poor's, and Fitch's results are by and large similar.

The Markov assumption certainly facilitates the description of firms' long-term default behavior considerably. It is, however, not clear how close it is to reality. The Markov assumption implies that there are no systematic consecutive downgrades or upgrades of one specific company over several years as could be observed with IBM in the 1980's or SAP in the 1990's. We, therefore, compare the above results with Standard and Poor's directly estimated 10-year cumulative default probability[156].

Rating	Standard and Poor's 10-year cumul. def. prob.	Wilson's results	Ratio
AAA	0,51%	1,74%	341%
AA	0,60%	4,12%	687%
A	1,17%	9,58%	819%
BBB	2,89%	22,63%	783%
BB	11,80%	46,75%	396%
B	31,85%	69,12%	217%
CCC	46,53%	83,58%	180%

Table 4: Directly estimated 10-year cumulative default probabilities

It turns out that apart from the AAA rating the deviations between the outcomes are even larger, with Wilson's results being on average 4,98 times larger than Standard and Poor's.

Part of this result seems to be due to the transform of speculative grade default probabilities in rating specific default probabilities in step 3 of the model. Figure 20 compares *n*-year conditional and unconditional default probabilities after a severe crisis for speculative grade companies. Albeit the strong persistence of shocks in the model[157], the 10-year conditional cumulative default probability is only about 10% larger than the unconditional toe for the same time horizon. This is considerably less than the deviations stated in Table 3 and Table 4 for rating specific cumulative default probabilities.

[156] Cf. Standard and Poor's, 2001, p. 16.

[157] See Figure 9 above. The process and the crisis scenario used for the analysis in Figure 20 is the same as described in footnote 140.

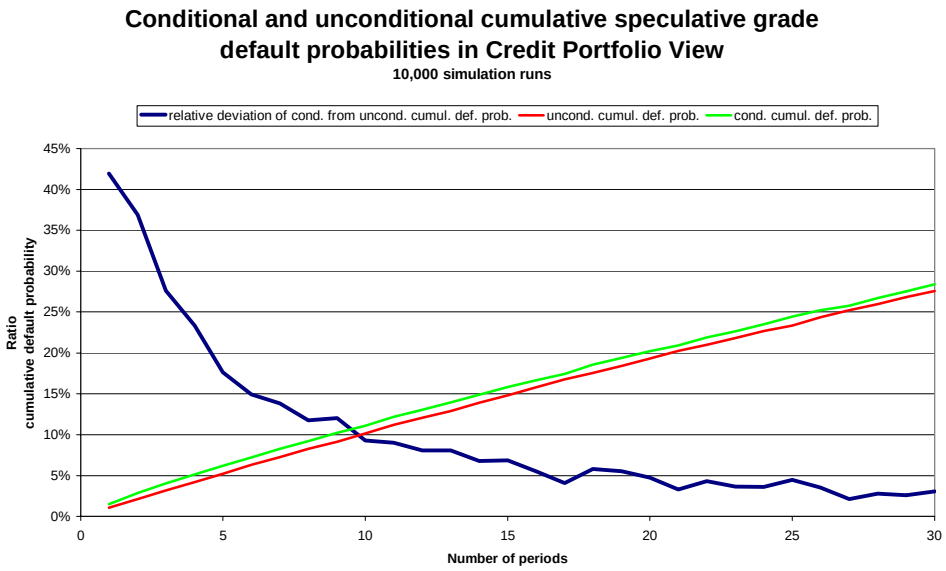

Figure 20: Conditional and unconditional cumulative default probabilities in Credit Portfolio View

(6) Conclusion

Wilson intended to empirically relate firms' default behavior to macroeconomic risk factors. However, we have found that Credit Portfolio View draws this link rather loosely and very indirectly. Moreover, in each of several steps to the final estimation result multiple problems and uncontrollable errors occurred. The current model does not solve the task and adds more noise to the analysis than explanatory power.

We, therefore, conclude that Credit Portfolio View can only be the beginning of the understanding of the empirical relationship between risk factors and default probabilities. However, some of the insufficiencies of the model can be alleviated. Especially the systematic estimation error in the logistic regression of default probabilities against macroeconomic factors can easily be cured by not applying a linear regression to transformed default rates, but by using non-linear logistic maximum-likelihood techniques. But deficiencies of the extrapolation of macro factors and of the derivation of transition probabilities remain.

7. Rating based estimation of default probabilities: the CRE model

The last approach to estimate default probabilities of rated companies we would like to present is part of the Credit Risk Evaluation model. The CRE model was developed by the author for the Center for Risk and Evaluation GmbH & Co. KG, Heidelberg.

a) Main concept and derivation of default probability

In order to properly depict the characteristic features of different customer segments, the CRE model is very flexible in that it allows the use of distinct estimation methods for distinct groups of clients, which includes firms in different industries and countries and private customers.

The CRE model comprehends three empirical influences of client's default probabilities: country risk and macro and micro economic influences on clients' default risk.

(1) Country risk

The default risk of the client's resident country is an essential factor in the assessment of default risk because in many cases firms and private customers are prevented from fulfilling their financial obligations against their own will by purely external reasons.

We define a country's default as any temporary interruption of money transfers from that country to the home country of the financial institution that makes the analysis. Disturbances of money transfers can be due to economical or political reasons such as wars and shortages of foreign money of the central bank. Numerous such crises could be observed since the 1960's in Eastern Europe, Asia, Latin and Middle America, and Africa[158].

If a financial institution does not make it's own investigation of country risk, country ratings and estimates of default probabilities can be obtained from the international rating agencies such as Moody's or Standard and Poor's[159].

[158] For instance in Argentina, Costa Rica, Iran, Ghana, Guatemala, Indonesia, Uganda, Nicaragua, Zaire, Yugoslavia, Nigeria, Panama, Rumania, Uruguay, and others. See UBS (2001), p. 6.

[159] Confer to UBS (2001), pp. 5f. Examples for Moody's and S&P's country ratings are

Rating	Country
AAA	Germany, France, USA, Switzerland, Austria
AA	Belgium, Italy, Spain
A	Greece, Israel
BBB	Croatia, Lithuania, Poland, South Africa, Uruguay
BB	Argentina, Columbia, Mexico
B	Brazil, Lebanon, Rumania, Turkey, Venezuela
CCC	Russia

Usually country risk is incorporated into credit risk models in the way that all clients that reside in a country are downgraded to the countries' rating. This means a company in Mexico (BB) cannot have an A rating, even if it is innovative, competitive, and financially well managed.

This methodology has two disadvantages. Firstly, it cannot further distinguish between clients who have the same rating as or a higher rating than their country. It is, however, not plausible that AAA, A, and BB counterparties in Mexico all have the same financial prospects and the same creditworthiness. Secondly, it does not express the effect that all of a country's residents are affected simultaneously by a war or a general financial crisis. This argument shows that country risk is not only important in terms of default probability, but that it is also a concept of dependence between counterparties.

For this reason, the state of a country, whether it is financially intact or in default, is treated as a background factor in the CRE model. A client's rating is considered as being conditional to his country being intact. Thus, there can be an A-rated company in a BB country such as Mexico. If, on the other hand, a country defaults, all clients residing in it automatically default, too. This setting can incorporate country risk, the differences in credit quality between risk grades, and the chain reaction declenched by a country default.

The home country of the financial institution that makes the analysis is a special case. This is, firstly, because its countrymen do not need foreign money to meet their obligations. The country's central bank plays no role in this financial relationship. Even if the central bank runs out of foreign money, clients can pay their liabilities. It is, secondly, because a political crisis such as a war would directly involve the financial institution, too. This is an exceptional situation that is not relevant for credit risk management. We, therefore, assume that the bank's home country is risk-free and cannot default.

Taking into consideration that countries themselves are strongly interrelated, this understanding can be extended to groups of countries that are closely fraternized with the bank's home country. It is, for instance, hardly imaginable that a country that is part of the eurozone can isolatedly default. Sharing the same currency and exchanging most of their imports and exports, it is likely that a crisis would quickly spread to the partner countries. For a respective bank it is, therefore, reasonable to consider the eurozone as its home country.

A segment of customers that needs special attention are big multinational companies that are well diversified among several countries. If a company has major dependences in other coun-

tries, it is entirely possible that it can still meet all of its international obligations even if its headquarters or another part of it are temporarily cut off. This effect is described in the CRE model by randomization. This means that if a country defaults occurs, it does not surely spread to the company, but just with a certain probability.

It is important to note that it is not obligatory to consider country risk in the CRE model. Especially small regional banks frequently do not have any or very little business abroad. In this setting, it does not seem necessary to monitor country risk.

(2) Micro economic influences on default risk

If a bank knows its clients' business relationships and their reasons for default well, it can take microeconomic influences on default risk into consideration.

Besides the general business risk, companies carry a distinct default risk similar to financial institutions. The default of a business partner can immediately bring a firm into financial distress. A prominent example is the evasion of the big construction company Schneider AG in Germany in 1995. Dozens of smaller construction companies that worked exclusively on the building sites of the Schneider AG would have had also been drawn into default if the Deutsche Bank, one of the major creditors of Schneider AG, had not acknowledged its social responsibility as the leading financial institution in Germany and paid the outstanding debt.

Financial distress can carry over from one company to another mainly for two reasons. Firstly, a firm can run into liquidity shortages if a relevant part of its short term outstanding claims cannot be expected to be paid in the near future, if at all. This is a major cause of bankruptcy in the eastern federal states in Germany. Secondly, a firm can loose part of its sales structure if an important client is lost. It is hit particularly hard, if the market is small and if it has only a few clients. Both setbacks were the case in the example above. This argument also shows that economic subjects do not only depend upon each other via systematic risk factors in the economy as is often said[160].

Micro economic influences on default risk can be a valuable tool for the modeling of default probabilities if a regional economic structure is dominated by only a few large companies. Industries in which this is frequently the case are the traditional sectors including the steel, coal, and car industries. For instance, Wolfsburg strongly depends upon Volkswagen, Clermont Ferrand upon Michelin, Longbridge upon Rover, and Bitterfeld upon Leuna and Buna. A collapse of the major company leads to the sudden unemployment of thousands in the af-

[160] Cf. Wilson (1997a,b), Credit Suisse Financial Products (1996)

fected region. Many of the dismissed workers will have problems to quickly find a new job so that mortgage loans and other credits are immediately endangered. Furthermore, many smaller companies in the area can now only sell fewer goods and might be drawn into the crises.

Note that similar micro economic relationships can also imply an improvement in a company's credit quality if a direct competitor defaults. This phenomenon, however, is probably of minor weight.

Besides general economic dependencies, an important case of relationships among individual counterparties is domination due to possession or close financial association. For instance, companies that belong to the same owner or to the same holding are likely to face financial difficulties if the holding or the owner defaults. Being obliged to report this kind of affiliation to the legal banking authorities, banks have well researched data at hand about this special situation[161].

Finally, clients often appear in various roles in a bank's portfolio. A client can be an obligor or a trade partner. In this case, his credit risk results from the transactions made with him directly. On the other hand, the bank can hold a position in a short put option on the client's equity. In this case, someone else would be the direct counterparty and the client under consideration is only indirectly involved. However, he originates a credit risk to the bank because the short put position would drastically increase in value if he defaults. For marginal risk analysis, pricing, and exposure limitation it makes a great difference whether a client has to pay and is made responsible for the credit risk that results from his only own transactions or for the total risk that is incurred by him as a result of direct and indirect financial interactions.

This problem can easily be solved if micro economic relationships are introduced into the model. Here, the client can be represented by two identical copies that fully depend upon each other, i.e. they either simultaneously default or survive. Both copies only differ in the amount of exposure assigned to them and in the stand alone probability of default. The first one corre-

[161] See for instance KWG § 19, 2: "1 Im Sinne der §§ 10, 13 bis 18 gelten als ein Kreditnehmer zwei oder mehr natürliche oder juristische Personen oder Personenhandelsgesellschaften, die insofern eine Einheit bilden, als eine von ihnen unmittelbar oder mittelbar beherrschenden Einfluß auf die andere oder die anderen ausüben kann, oder die ohne Vorliegen eines solchen Beherrschungsverhältnisses als Risikoeinheit anzusehen sind, da die zwischen ihnen bestehenden Abhängigkeiten es wahrscheinlich erscheinen lassen, daß, wenn einer dieser Kreditnehmer in finanzielle Schwierigkeiten gerät, dies auch bei den anderen zu Zahlungsschwierigkeiten führt. 2 Dies ist insbesondere der Fall bei:

- allen Unternehmen, die demselben Konzern angehören oder durch Verträge verbunden sind, die vorsehen, daß ein Unternehmen verpflichtet ist, seinen ganzen Gewinn an ein anderes abzuführen, sowie in Mehrheitsbesitz stehenden Unternehmen und den an ihnen mit Mehrheit beteiligten Unternehmen oder Personen, (...)

- Personenhandelsgesellschaften und jedem persönlich haftenden Gesellschafter sowie Partnerschaften und jedem Partner und

- Personen und Unternehmen, für deren Rechnung Kredit aufgenommen wird, und denjenigen, die diesen Kredit im eigenen Namen aufnehmen. "

sponds to the business done with the client personally, the second stands for the indirect credit risk he causes. Hence, both sources of risk will remain separated in all subsequent risk management actions.

Micro economic relationships are modeled by randomization in the CRE model. Given the default of a client, his business partners fail with certain probabilities. Randomizing weights can be freely chosen. They can be directly estimated from the bank's default experience.

Micro economic dependencies between counterparties are typically asymmetric. An employee usually depends much more on his employer than vice versa. The craftsman is severely affected by the default of the large construction company, while the construction company is most likely almost independent from the smaller firms that work for it.

It is worth mentioning again that micro economic influences on default risk do not have to be considered if they are not relevant for the bank's market segment or if the bank cannot supply the necessary data.

(3) Macroeconomic influences on default risk

The two influences on default probabilities previously discussed made no explicit use of a client's rating. Nevertheless, the rating is a decisive information for the assessment of a counterparty's future default behavior. In the following, we assume that known cases of default due to country risk or micro economic relations are left out of the estimation of rating based default probabilities.

As we already indicated in Figure 8 and the discussion of Wilson's model, the macroeconomic environment has a significant influence on a portfolio's default situation. However, the impact of the current macroeconomic scenario is not the same in all industries and especially not in all rating grades and customer segments. For instance, a default of a AAA-company has never been observed within a time horizon of one year, not even in the darkest economic crisis. For this reason, it is impossible to directly estimate systematic influences on the default probabilities of high grade companies.

Another example is private customers who are not self-employed. They are only indirectly affected by macroeconomic shocks via their employer. Only if the employing company collapses or if the employee is dismissed, does the economic environment reach the private customer.

Consequently, it should be possible to not take account of macro factors in the evaluation of the default behavior of certain customer segments. In the CRE model, the method used to estimate default probabilities can, therefore, be independently chosen for each group of clients. Customer segments can be defined freely by the bank. For high grade firms, for industries that are not closely integrated into the macroeconomic environment, and for private customers the mean value model can be used.

In the following, we would like to present the macro model that can be employed in the CRE model to assess default probabilities of speculative grade firms. The term 'macro model' may be misleading, though, since the CRE model does not contain an explicit prognosis component for the development of macroeconomic factors. The forecast of macro factors is a non-trivial pursuit. Year by year, many highly profiled research institutions in various countries try in vain to make precise statements about the future unemployment rate, the GDP growth rate etc. Any attempt to include conjectures about future values of macro factors into a credit risk model needs to be simplistic and, in fact, misleading.

For this reason, the CRE model employs only values of macro factors that are already observable at the evaluation time to estimate firms' future default behavior. This approach is in line with most analyses in the literature, which found that the economic environment influences the economy's default situation with a certain time lag[162]. Hence, the macro data known today is sufficient to predict a firm's default behavior over the next 1-3 years. However, it is hardly possible to make forecasts over longer time horizons. Therefore, the CRE model uses macro factors only to make short-term predictions of default rates[163].

As already mentioned, the impact of the business cycle on firm's default rates not only depends on their belonging to a certain industry, but also upon their rating. For this reason, the CRE model does not comprise all speculative grade companies in an industry to a 'credit risk indicator' as in Wilson (1997a) or in Kim (1999) because the inference of rating specific default probabilities from the indicator is not possible without major additional imprecisions[164]. A default has a much greater impact on a client's or a portfolio's credit risk than a simple rating downgrade that leads to a comparably small change in value. Thus, imprecisions particularly need to be avoided in the estimation of default probabilities. The regression of default

[162] Cf. the discussion in Bär (2000), Lehment et al. (1997) and others.

[163] Long-term prognoses of default probabilities are vague and unreliable at any rate. A simple method to extrapolate one-year default probabilities to n-year time horizons is to assume defaults to follow a Markov process and multiply the transition matrix n-times with itself.

[164] See the discussion of Wilson's model above.

rates against macro factors is, therefore, done for each industry and each speculative grade rating category separately.

To relate systematic economic factors to firm's default rates, we propose the following simple model. Let $Y_i, i = 1,...,n$ be the observed default rates in period *i*. Let $X_{1i},...,X_{mi}$ be the macroeconomic factors explaining the default situation in period *i*. Taking account of a certain time lag, $X_{1i},...,X_{mi}$ can be observations made in period *i* - 1 or earlier. The model is then described by the equation

$$Y_i = \beta_0 + \beta_1 X_{1i} + \ldots + \beta_m X_{mi} + \varepsilon_{Yi}$$

where ε_{Yi} is an error term with mean zero and unknown, but constant variance σ. We do not make any distributional assumptions concerning ε_{Yi}.

The parameters $\beta_0,\ldots,\beta_m$ can be estimated using least squares techniques. This model is equivalent to the mean value model if $\beta_1,\ldots,\beta_m$ are identically set to zero. All parameter values can be estimated consistently[165].

An important feature of this model is that it does not require the independence of defaults conditional to a macroeconomic setting[166]. This is consistent with the larger architecture of the CRE model since we will assume that sectors can be correlated beyond the macroeconomic influences on default probabilities[167]. This is a significant advantage with respect to Credit Risk + where firms in different sectors are necessarily independent and to Credit Portfolio View where sectors are also independent if the error terms are uncorrelated[168].

Due to the direct linear regression, extrapolated values for Y_i could respectively be negative or outside the unit interval. In practice this is not an urgent problem, though, since we don't intend to simulate error terms. Moreover, it is straightforward to confine Y_i to reasonable values by introducing upper and lower bounds, i.e. if the extrapolated value is negative or below a certain threshold, the expected default probability is set to the minimal value that is thought acceptable. However, if the current macroeconomic setting implies an extreme deviation of estimated default probabilities from their long term mean, this result should be handled with care because it could indicate that a structural break has occurred and that the validity of the model needs to be put into question.

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165 While the least squares estimator is consistent, it is not necessarily efficient because the error terms are non-normal and can also be skewed if correlations are positive. Further research has to be done to reduce the variance of the estimation.

166 This assumption was essential in Credit Portfolio View and Credit Risk+. Without it all estimations are wrong in these models.

167 See section II.A.2 below.

168 A problem which is not discussed in Wilson's articles.

Instead of upper and lower bounds, a probit or logit regression could be used to assure estimated default probabilities lie between 0 and 1. These models can be consistently estimated by non-linear techniques such as maximum-likelihood[169].

(4) Example

To give an impression of the quantitative features of the model, we take an example. We assume default probabilities to depend upon one macroeconomic factor, i.e. we have

$$Y_t = \beta_0 + \beta_1 X_t + \varepsilon_{Yt}.$$

Parameters are chosen as $\beta_0 = 0.51\%$, and $\beta_1 = 0.25\%$. The distribution of the error term is not supposed to follow a specific distribution. Historic realizations of the macro factor are described by the same AR(2)-process used for the analysis of Credit Portfolio View above, i.e. by

$$X_t = 0.4X_{t-1} + 0.4X_{t-2} + 0.1\varepsilon_{Xt}$$

where ε_X is standard normally distributed. Note that the shape of the macro process is not used for any of the estimations in the CRE model. We assume clients in the segment under consideration to have a correlation of ρ.[170] ρ varies between 0 and 90%. Estimations were performed with different numbers of clients in the segment and different numbers of periods of data.

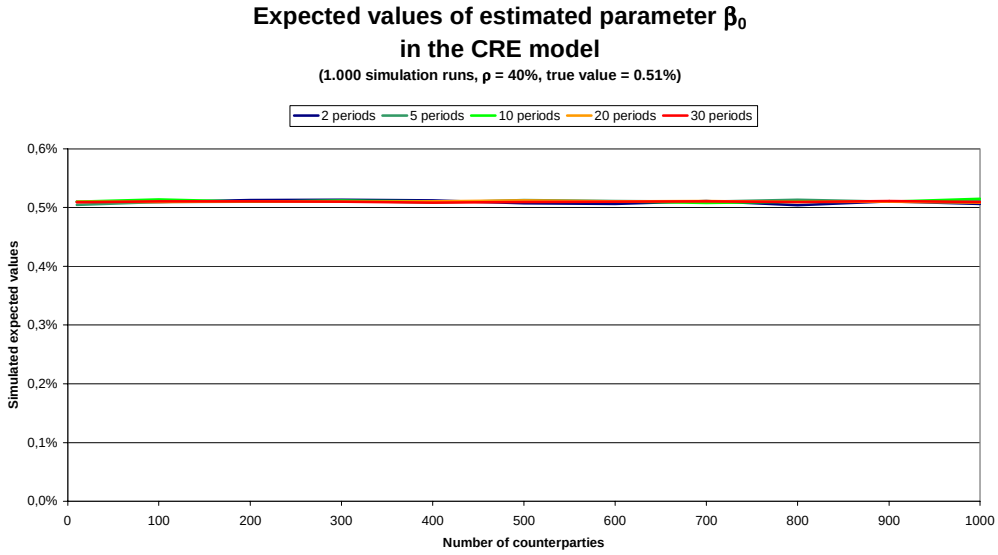

Figure 21: Expected values of estimated parameter β_0 in the CRE model under correlated defaults

[169] Cf. to Maddala (1983), pp. 25ff.

[170] For the concept of correlation in this context, confer to section II.A.2 below.

Figure 21 shows the expected estimation results of the mean-value-parameter β_0. It is obvious that the parameter is unbiasedly estimated for any number of counterparties and periods, although defaults are not supposed to be independent, but significantly correlated.

The standard deviation of the estimated parameter value, on the other hand, does depend on the number of clients and especially on the number of periods of data available, as is to be expected. However, the informational content in the number of counterparties is quickly exploited.

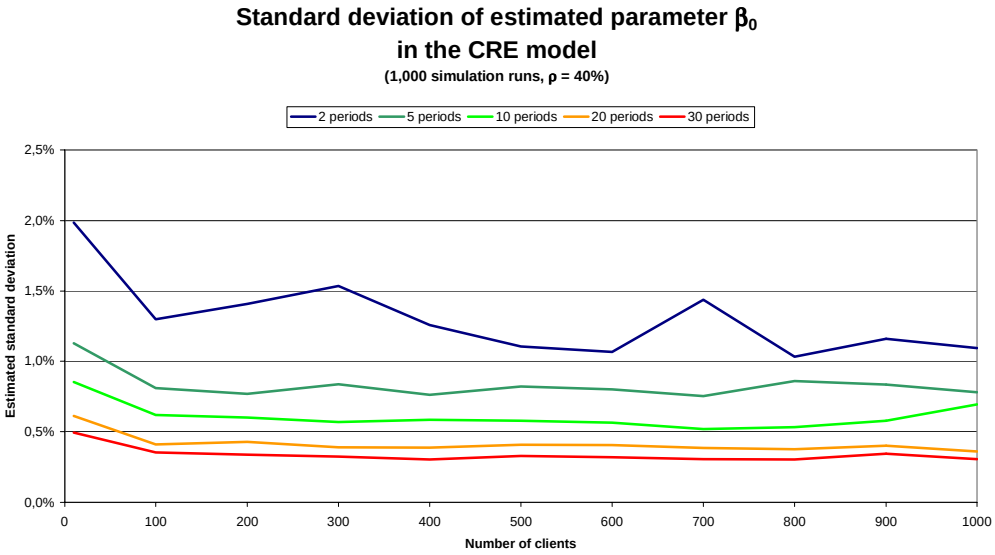

Figure 22: Standard deviation of estimated parameter β_0 in the CRE model under correlated defaults

The same observations can be made when we look at the volatility parameter β_1.

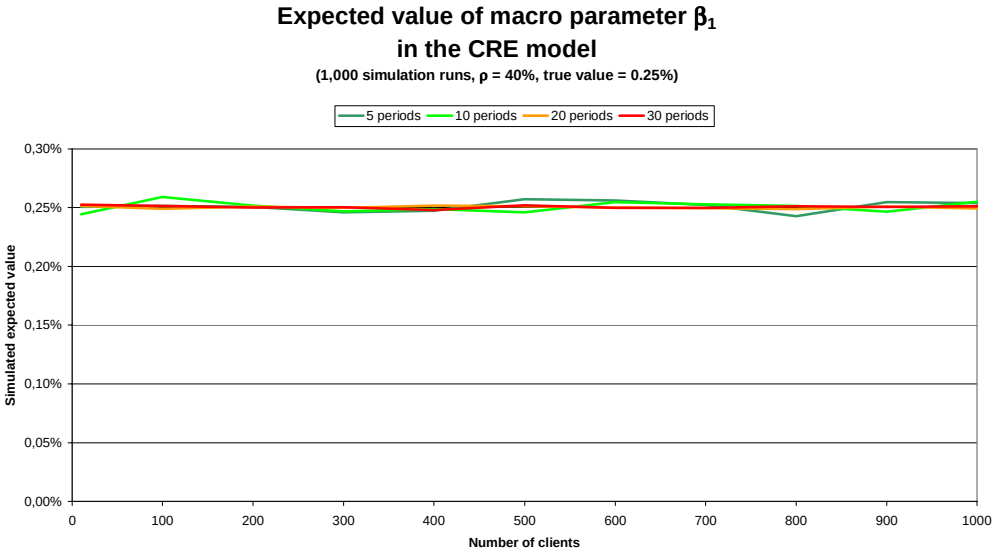

Figure 23: Expected value of macro parameter β_1 in the CRE model under correlated defaults

The estimation is unbiased for any number of clients and periods even if defaults are correlated. Again the parameters standard deviation predominantly depends on the number of periods.

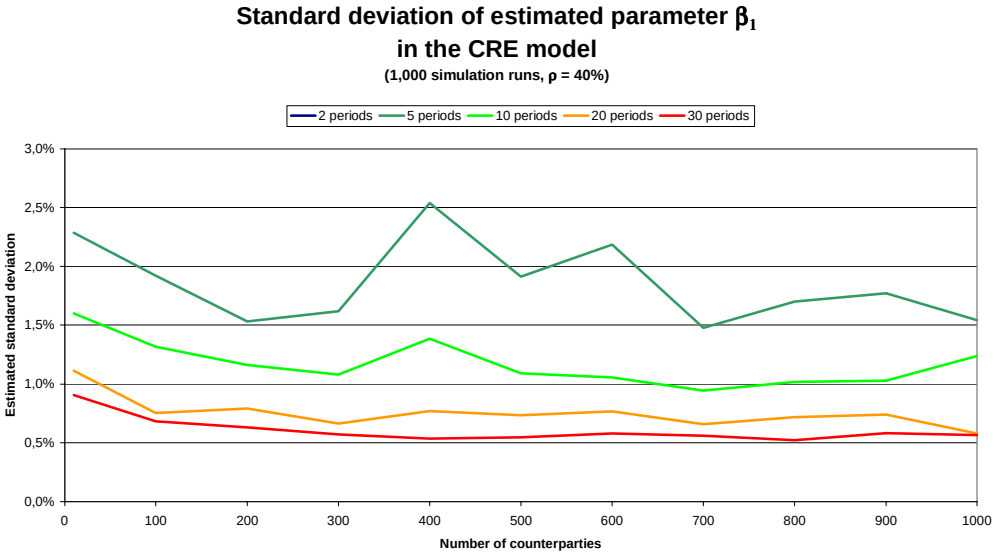

Figure 24: Standard deviation of estimated parameter β_1 in the CRE model under correlated defaults

The estimator's standard deviations are not only related to the number of counterparties and periods as can be seen in the charts, but also to the degree of correlation among counterparties. As in the mean value model, standard deviations are relatively large compared to the size of the respective parameters and increasing in the correlation.

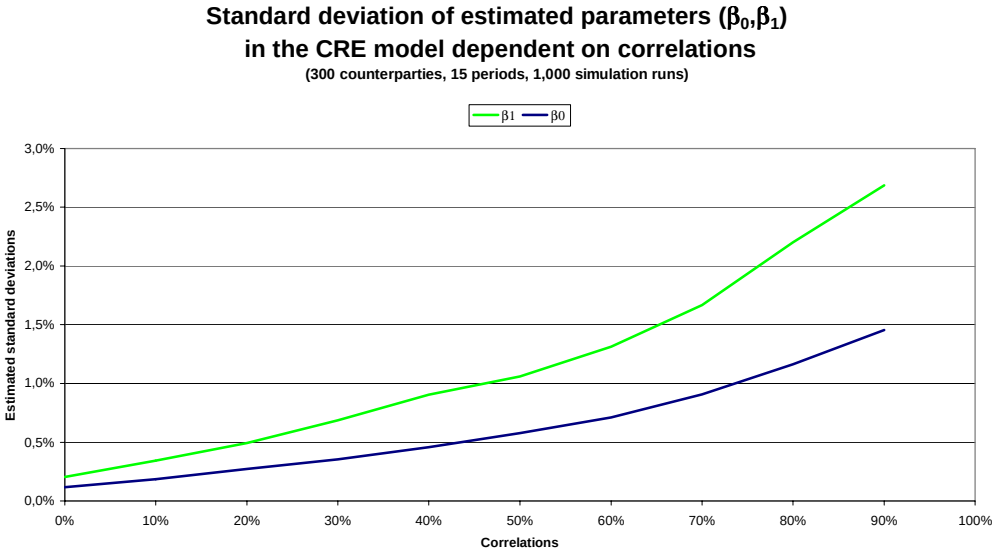

Figure 25: Standard deviation of estimated parameters (β_0,β_1) in the CRE model dependent on the size of correlations

Figure 26 shows the joint distributions of parameters β_0 and β_1, clearly illustrating the increasing variability and skewness in the estimation for high correlations[171].

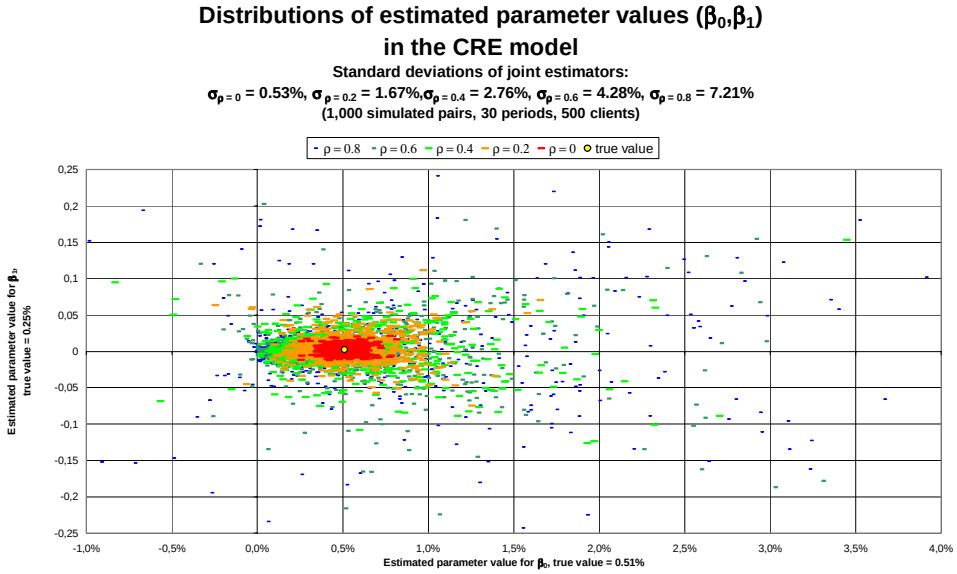

Figure 26: Distributions of estimated parameter values (β_0,β_1) in the CRE model under various correlations

Table 5 sums up the main characteristics of the distributions of parameter estimators[172]. The similarities to the results in the mean value model are striking. Here again the mean parameter value is unbiasedly estimated while the uncertainty and also the skewness of the estimation results increases with correlations. This result holds for both parameters β_0 and β_1.

β0	ρ = 0	ρ = 0.2	ρ = 0.4	ρ = 0.6	ρ = 0.8
Mean	0,511%	0,514%	0,513%	0,508%	0,527%
Std-Dev	0,072%	0,211%	0,383%	0,558%	1,018%
Skewness	9,10E-11	9,05E-09	1,10E-07	5,30E-07	6,06E-06
Kurtosis	1,01E-12	1,01E-10	2,23E-09	2,25E-08	8,43E-07
β1					
Mean	0,251%	0,259%	0,245%	0,256%	0,260%
Std-Dev	0,527%	1,656%	2,738%	4,227%	7,140%
Skewness	2,29E-08	3,77E-06	2,62E-05	2,25E-04	5,56E-04
Kurtosis	2,50E-09	5,47E-07	1,17E-05	9,90E-05	7,32E-04
500 clients	**30 periods**	**1,000 simulation runs**			

Table 5: Characteristics of distributions of parameter estimators (β_0,β_1) in the CRE model under various correlations

[171] See also Table 5 below.

[172] It should be noted that Table 5 corresponds to Figure 26 and, hence, includes only 1,000 simulation runs. Thus, imprecisions due to sampling errors may occur, while the general trend seems to be clear.

(5) Conditional transition probabilities

The last step in the macro model within the CRE model is the derivation of transition probabilities conditional to the current economic background. Since we have done the estimation of conditional default probabilities not only for different industries, but also for individual rating grades, we only have to calculate non-default transition probabilities[173].

Similar to Wilson's approach, we base the derivation of transition probabilities on the ratio of the estimated conditional default probability to its long-term mean. Precisely, we suggest a linear transformation of the following form

$$\hat{p}_{tj} = w_j \left(\frac{\hat{p}_{tn}}{\overline{p_n}} - 1 \right) + \overline{p_j}$$

for $j = 1,\ldots,n$, where n is the number of rating grades, $\overline{p_n}$ is the estimated long-term mean default probability of the rating grade under consideration, $\hat{p}_{tn}$ is the estimated conditional default probability, $\overline{p_j}$ is the estimated long-term mean probability to migrate from the present rating to rating j, $\hat{p}_{tj}$ is the estimated conditional transition probability, and w_j is a sensitivity weight.

Since we have already estimated $\hat{p}_{tn}$, we choose $w_n = \overline{p_n}$. The other parameters w_j for $j = 1,\ldots,n-1$, have to be estimated from the bank's experience. They could, for instance, be chosen as the least squares estimators of

$$\sum_{i=1}^{m} \left(\pi_{ij} - w_j \left(\frac{\hat{p}_{in}}{\overline{p_n}} - 1 \right) + \overline{p_j} \right)^2 \overset{!}{=} \min_{w_j \in \mathbf{R}}$$

for grades $j = 1,\ldots,n-1$, where π_{ij} is the observed transition frequency to rating j in period i for $i = 1,\ldots,m$ if there are m periods of data available, and $\hat{p}_{in}$ is the default probability for period i interpolated from the estimated values for β_0 and β_1 above and the observed macroeconomic factors. Note that for any value of w_j the mean of the resulting estimated transition probabilities is equal to $\overline{p_j}$. This is due to the linear regression model of $\hat{p}_{tn}$ where the mean of the estimated default probabilities is always equal to the mean of the observed default frequencies[174].

[173] Credit Portfolio View, on the other hand, has to derive individual default and transition probabilities from the pool of speculative grade firms in an industry for all rating grades.

[174] $\overline{Y} = \beta_0 + \beta_1 \overline{X}$ is one of the equations to be solved in the least squares estimation.

It is not necessary, but intuitive that the weights w_j should be negative for ratings better and be positive for ratings worse than the present one. This is because a rating upgrade should be less likely and a downgrade more likely than the average during a recession.

It is, however, possible that estimated future transition probabilities $\hat{p}_{m+1,j}$ are negative, larger than one or that their sum does not add up to one. Therefore, we introduce 0 and 1 as the upper and lower bounds for the estimated values[175] and then standardize them by

$$\hat{\hat{p}}_{m+1,j} = \frac{\hat{p}_{m+1,j}}{\sum_{k=1}^{n-1} \hat{p}_{m+1,k}} \left(1 - \hat{p}_{m+1,n}\right)$$

for grades $j = 1,\ldots,n-1$.

C. Exposures

Having assessed transition and default probabilities, the next step in credit risk analysis is the judgment of how high losses could possibly be if a credit event such as a counterparty's default or a rating migration occurred. Dealing with the chance of recoveries in the next section, we define the term of exposure as the amount of money that is subject to credit risk, i.e. we assume recoveries to be zero to calculate a transaction's exposure.

The potential losses and the bank's exposure to risk are not selfexplanatory, but depend upon a number of influence factors such as the type of credit event, the precise definition of exposure, the type of financial instrument traded with a client, the level of aggregation of exposure among clients, the risk management purpose exposures are calculated for, and the position in the life cycle of a transaction. We will discuss each topic and give examples.

a) Roles of counterparties

It is worth noting that clients can affect a bank's credit risk in various ways depending on the role they play in the bank's financial relationships.

The most common function a counterparty can have is the one of an obligor or direct trade partner. In this case, it is selfexplanatory that all business done with this client is directly affected and might change in value if he undergoes a credit event.

[175] I.e. if an estimated migration probability is negative, it is set to zero etc.

However, an economic subject can also be indirectly related to a bank's portfolio, for instance as an issuer of stock or bonds if the bank has bought or sold derivatives that have these securities as an underlying asset. Here it might happen that the direct counterparty has a steady credit quality while the value of the derivative considerably alters due to a deterioration or improvement in the credit worthiness of the issuer of the underlying asset.

Especially short put options on stock and bonds carry an extreme credit risk which is attached to the indirectly involved counterparty. Put options experience their maximum increase in value if the value of the underlying asset breaks down. In the case of a default of the counterparty, a put is sure to be exercised. Hence, a future default of the issuer of the underlying incurs an (additional) loss that is equal to the actual value of the underlying asset.

This shows that the often heard statement that liabilities are not subject to credit risk is not true since it may happen that the size of a liability changes owing to a credit event that might be caused by an indirect counterparty. The indirect counterparty risk even makes up for 80-90% of most banks' default risk in trade portfolios. However, always being a liability towards the direct counterparty, a short put position does not carry a credit risk with regard to him because his default would not protect the bank from having to pay its obligations if his position is taken by one of his creditors.

This is different in case of long option positions. Here we have two sources of credit risk, one due to the direct counterparty as long positions are always assets and the second one due to a deterioration in credit quality of the issuer of the underlying asset which can also be referred to as a special interest rate or a special equity price risk.[176]

Although tied to the same probability of default, both types of counterparty risk should be well distinguished in credit risk management because a client has in general no influence on the size of the exposure or risk that he entails as an indirect counterparty. If both types of risk are mixed, his credit line, for instance, could quickly be used up for no reason apparent to him.

b) Concepts of exposure

A major complication in the calculation of exposures is to find a concept that applies to all kinds of financial products consistently and can be used to express the consequences of a default.

[176] See for instance Grundsatz 1, § 23 and § 25.

Traditionally, credit products such as bonds and loans were accounted for by their book value and derivatives by their notional. Both concepts have two fundamental disadvantages. Firstly, they are not related to the actual market situation. Changes in market prices and interest rates do not affect a swaps principal or a loans book value. They may, however, significantly influence their market value and their replacement costs. Secondly, both definitions tend to overstate the instruments' true potential loss. This is especially striking for derivative products where the notional often has no direct relationship to a contract's value.

A first guess for a better measure of exposure in case of default would be the trade's

(1) Present value

A deal's present value[177] has the advantage that it is no longer static and keeps track of the development of market prices. However, some derivative positions such as swaps, FRAs, futures and short option positions can take on negative values. As already mentioned above, liabilities are not affected by the default risk of the direct counterparty. This means that trades with negative present values are not in danger of being lost because a creditor in default would hand the claim on to his own creditors. Hence, we render the measure of exposure more precise and define the

(2) Current exposure

A transaction's current exposure C is the positive part of its present value V, thus,

$$C = \max(0,V).$$

The current exposure is generally used for fixed income products that are relatively stable in value such as loans and bonds. In addition to that it has many applications in the management of spot and short term credit risks where no great changes in present value are to be expected also for more volatile transactions as, for instance, the replacement risk for newly closed trades.

The replacement risk[178] is the risk that a transaction that has been agreed upon will not be realized. No party has made any payments. In the case that the transaction is canceled, the resulting loss is equal to the costs caused if the same trade has to be closed again. This is the

[177] Although only approximately equal, we use the terms of present value and market value synonymously throughout this chapter. Especially in the examples we will neglect the transactions' default risk. This leads to a slight overstatement of the resulting exposures. It seems to be a suitable simplification, though, because, being merely an input for limitation, equity allocation, and portfolio management, exposures do not need that degree of precision that is required for pricing, for instance.

[178] This risk is also known as "Wiedereindeckungsrisiko".

difference between the transactions current present value and the agreed price. Thus, if the transaction has risen in value, the exposure is positive and it is zero otherwise.

Similar to the replacement risk is the risk that occurs in the same setting if payments had already been made and the delivery of the security remains outstanding[179]. Here the exposure is equal to the present value of the outstanding delivery[180].

The third spot risk where the concept of current exposure can be applied is the risk that payments or deliveries that are due do not occur while the own party has not taken any action yet[181]. It is identical to the replacement risk with the difference that it includes transactions that have already been taken[182].

(3) Examples

To illustrate the notions of present value and current exposure, take a long equity call option position and a long FRA position as examples.

Let us suppose the equity call to mature in 1 year with a strike price of 100 €. The one-year risk free interest rate is equal to 6% and the actual equity price is 80 € with an annual volatility of 1. Then, following the Black-Scholes analysis, the option's present value is equal to

$$V_{option} = 26.67 \text{ €}$$

and, thus, the current exposure is as well. Note that a long option position is a right to something without any obligation once the option premium has been paid. In such a case, the present value and the current exposure are always identical.

This is different in case of a forward rate agreement (FRA). A long position in a FRA is the agreement to grant a credit at fixed conditions at some point in the future. Apparently, the sign of its market value depends on the market conditions. If interest rates go down, a long FRA position has a positive market value. If interest rate rise, its value is negative.

Let us again suppose the one-year interest rate is equal to 6% and assume that the interest rate for 15 months is equal to 6.5% in continuous compounding. The FRA was closed 6 months ago and is the agreement to grant a loan of 100,000 € over a period of three months beginning in one year from now at the rate of 8.2%. Thus, its present value is equal to

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179 In German this risk is also called "Vorleistungsrisiko".

180 Cf. Grundsatz 1, § 27.1.2.

181 In German this risk is also called "Abwicklungsrisiko".

182 Cf. Grundsatz 1, § 27.1.1.

$$V_{FRA} = 100{,}000\left(e^{8.2\%\cdot 0.25 - 6.5\%\cdot 1.25} - e^{-6\%\cdot 1}\right) = -70.61€$$

and its current exposure is zero.

(4) Potential exposure

While derivatives are relatively stable in value over short terms, this may not be the case if longer time horizons are considered, which is usually the case in credit risk management. For this reason, the concept of current exposure is found unsatisfactory for applications such as portfolio management and exposure limitation. In terms of credit risk it is particularly problematic if a counterparty defaults when a transaction or a portfolio performs well in value due to favorable market conditions. Thus, the credit risk increases in times of *hausse* and decreases during a *baisse,* and, consequently, concepts of potential exposure[183] try to capture the effects of positive market behavior.

(5) Potential exposure of individual transactions or peak exposure

A typical definition of a position's potential or peak exposure is a possible value that will not be exceeded with a probability of say 95%, i.e. the transaction's potential exposure is an upper percentile of its value. It can be found by simulation of its underlying market factors or by analytic formulas depending on the trade's structure.

(6) Examples

If we consider the equity option already mentioned above, it is implied by the risk neutral valuation argument and the general distributional assumptions in Black-Scholes' theory that the equity value is lognormally distributed with

$$\ln(S_T) \sim \mathrm{N}\left(\ln(S_0) + \left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

where S_t is the equity value at time t, r is the risk free interest rate, and σ is the stock's annual volatility.

Thus, the option's potential exposure follows from the normal distributions percentile. The 95%-percentile, for instance, is equivalent to a deviation of 1.64 standard deviations from the mean. Hence, we have a potential exposure of

[183] Notions of potential exposure are often referred to as 'credit equivalents'.

$$P_{option} = e^{-r(T-t)}\left(\exp\left(\ln(S_0)+\left(r-\frac{\sigma^2}{2}\right)T+1.64\cdot\sigma\sqrt{T}\right)-X\right)=157.18\text{€}$$

for the option, where *X* is the strike price. Note that this value is 5.89 times the option's present value or current exposure.

To calculate the 95%-potential exposure of the long FRA position, we make the distributional assumptions of Black's interest rate model, i.e. we suppose the underlying forward rate to be lognormally distributed with

$$r_T^f \sim N\left(\ln r_0^f - \frac{\sigma^2}{2}T, \sigma^2 T\right)$$

where r_t^f is the underlying forward rate at time *t*, *T = 1* is the time to the granting of the credit, and σ is its annual volatility. We assume σ = *0.05*. Since the value of the long FRA position is increasing if the interest rates decrease, the upper percentile of the value of the FRA is equal to the respective lower percentile of the forward rate. Thus, we have for the 95%-percentile of the forward rate

$$r_T^f = \exp\left(\ln r_0^f - \frac{\sigma^2}{2}T - 1.64\cdot\sigma\sqrt{T}\right)=7.82\%$$

implying a 'long term' interest rate of

$$r_2 = \frac{r_1 T + r_T^f \cdot\left(T^* - T\right)}{T^*}=6.37\%$$

with T^* being the maturity of the future credit, and a potential exposure for the FRA of

$$P_{FRA} = 100{,}000\left(e^{8.2\%\cdot 0.25 - 6.37\%\cdot 1.25} - e^{-6\%\cdot 1}\right)=89.71\text{€}.$$

Four things are worth noting. Firstly, the FRA's potential or peak exposure is positive although its present value is largely negative and its current exposure is zero. This clearly shows that the definition of potential exposure as a percentile of the assets' value distributions somehow captures extreme market movements. The concept of peak exposure can find a credit risk even though we expect losses most of the time due to market risk, in our example with a probability of 88.2%.

Secondly and similarly, the notion of potential exposure tends strongly to the opposite direction from market risk analysis. If it is used for capital adequacy purposes, it is very likely that

too much equity is supplied if market and credit risk are added because they cannot always occur simultaneously by definition of the amount at stake.

Thirdly, the example of the forward rate agreement shows that the calculation of potential exposure requires more complicated models than the mere valuation of assets. Since the value of a FRA only depends on expected interest rates which are implied by the spot interest rate curve, its valuation can already be done if these elementary data are known. The potential exposure, however, results from a statement about how far future market factors might move away from their present position. Therefore, it demands distributional assumptions.

Fourthly, as indicated by the example of the equity option, a transaction's peak exposure can be relatively large compared to its current exposure even if the current exposure is already positive, in the illustration it was more than five times the size. Thus, the sum of all potential exposures can be considerable put side by side with the value of the respective portfolio. Moreover, since all transactions are treated isolatedly, impossible combinations can occur, e.g. a put and a call option that depend upon the same underlying asset move inversely in value. The potential exposure of the sum of both will, therefore, generally tend to be smaller than the sum of the potential exposures of the individual transactions.

These arguments imply that the concept of potential exposure should not be used in portfolio management in order to avoid an unnecessary exaggeration of the amount at stake. It can, however, be used in exposure limitation where diversification within a portfolio is not of interest and where the diversity of financial products in a client's subportfolio can be supposed to be small since banks usually do not hedge a portfolio's market risk on a client level. But even here the overstatement of the bank's exposition to risk would be considerable if exposures are aggregated over several levels up to the full portfolio.

(7) Potential exposure on a portfolio level

A much more moderate concept of potential exposure can be obtained if transactions' changes in exposition are calculated on a portfolio level. The calculation proceeds in several steps:

1. The total portfolio is segregated into subportfolios containing transactions with a relatively homogenous time to maturity[184].
2. The joint development of underlying market factors is simulated.

[184] The accuracy of the calculation can be increased if the portfolio is further subdivided by the type of the market factors, i.e. equity products are separated by interest rate products etc.

3. After each simulation run all trades' individual current exposures are calculated and added resulting in a distribution of current exposures for the respective subportfolio.

4. The excess of the desired percentile π_α of the exposure distribution over the subportfolios current exposure $C_{Portfolio}$ to the sum $\Sigma_{Principals}$ of all transactions' principals times the individual transaction's principal $\theta_{Principal}$ plus the transaction's present value V_θ is the estimator of the transaction's potential exposure P_θ if it is positive. Otherwise it is zero.

$$P_\theta = \max\left(\underbrace{\underbrace{\frac{\pi_\alpha - C_{Portfolio}}{\Sigma_{Principals}}}_{Add-on\ factor} \cdot \underbrace{\theta_{Principal}}_{Add-on\ basis} + V_\theta}_{Add-on}, 0 \right)$$

For high percentiles π_α it can be assumed that the add-on factor is positive. Sometimes the transaction's present value V_θ is replaced by its current exposure C_θ as the add-on basis assuring that the sum of add-on and C_θ is positive so that the max-function is not needed[185]. The add-on basis $\theta_{Principal}$ varies depending on the type of the trade. For a long option position it is the underlying asset to be received or delivered, for a swap it is the notional of the fictitious underlying bond, etc..

(8) Example

As a simple example to illustrate the concept, we take a portfolio consisting of 100 long equity call options on 100 different shares having a strike of 100 € and a time to maturity of 1 year. We assume that each share presently is traded at 80 € and has an annual volatility of 1. All shares' returns are lognormally distributed with a homogenous correlation of ρ.

[185] This is, for instance, the case in the legal exposure calculation method prescribed by the banking supervision authorities, cf. Grundsatz 1 § 10.

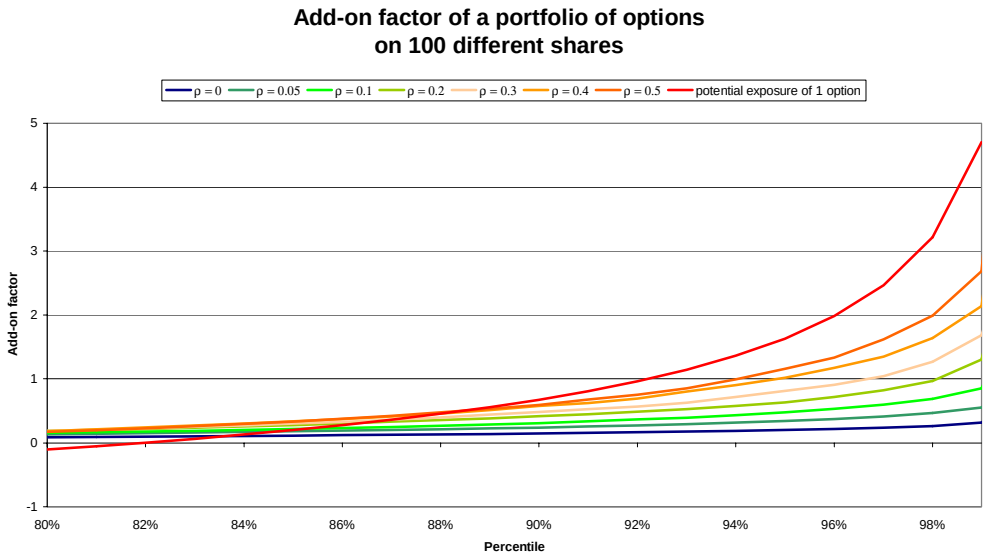


Figure 27: Add-on factor of a portfolio of options on 100 different shares

Figure 27 shows the resulting add-on factors for this portfolio for different levels of correlation. For high percentiles, the add-on factor increases with the correlation among shares and converges to the factor of an isolated option as ρ goes to one.

Figure 28 illustrates the effect of the add-on factor on the corresponding potential exposures. Even if correlations are high, the 95%-potential exposure is significantly lower if it is calculated on a portfolio basis. For instance, for $\rho = 40\%$ it is 31.4% lower, for $\rho = 10\%$ it is 58.8% less and eventually for $\rho = 0$ the reduction is as much as 73.3%.

Note that our example-portfolio is not overly realistic. For more heterogeneous, diversified, and especially for better hedged portfolios the effect is supposedly even more substantial.

Even though the methodology seems complicated at first sight, its practical implementation is much simpler because most banks use a similar technology and segregation of the total portfolio into subportfolios in market risk management. The concept of portfolio potential exposure, therefore, is rather a new interpretation of already existing data.

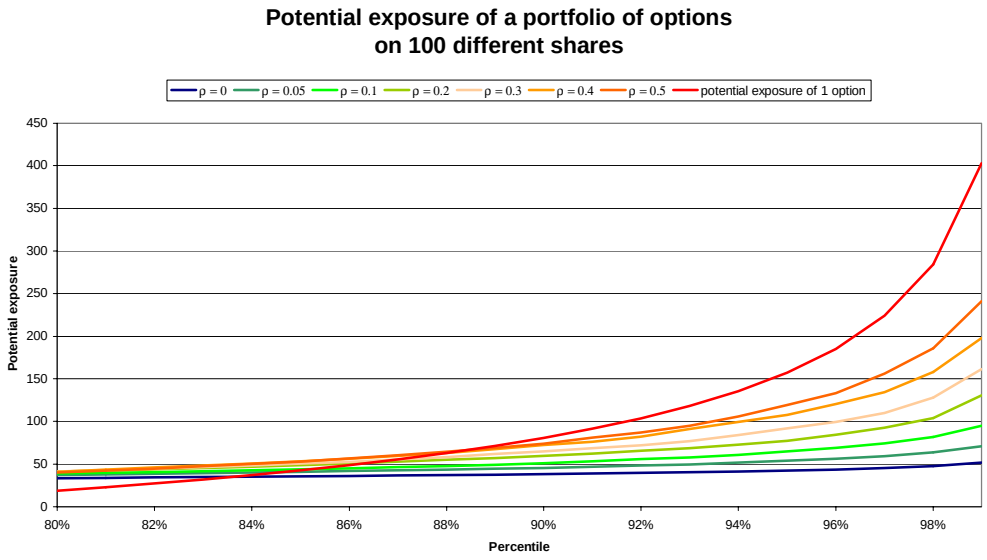


Figure 28: Potential exposure of a portfolio of options on 100 different shares

(9) Mean expected exposure

Another exposure type is called mean expected exposure. It is defined as the mean of the expected current exposure of a transaction until maturity or

$$M = \frac{1}{T}\int_{t=0}^{T} \max(\mathbf{E}(V(t)),0)dt$$

where T is the time to maturity, $V(t)$ the trade's present value at time t, and $\mathbf{E}$ the expectation operator. Instead of future market movements the mean expected exposure puts the focus on the time value of a transaction.

Options, for instance, have a negative time value if market conditions remain unchanged. A long equity call as in the example above only has a mean expected exposure of 15.64 € compared to a present value of 26.67 €, implying that the mean expected exposure is smaller that the current exposure. This does not seem convincing for most applications since options and similar derivatives are explicitly not made for stable markets.

However, the concept of mean expected exposure can well be used for fixed income products such as bonds, loans, or swaps. This can best be demonstrated by an example. Let's assume a flat interest rate curve at 6% and a step-up bond that also pays 6% p.a.. The bonds current principal is 500,000 € stepping up to 1,000,000 € in 6 months. Principal plus interest will be repaid in 1 year.

As the bond's interest is conform to the market, it is exactly worth par. Thus, its present value is equal to

$$V = 500{,}000\ €.$$

The mean expected exposure can be calculated as

$$M = \int_{t=0}^{1/2} 500{,}000\,dt + \int_{t=1/2}^{1} 1{,}000{,}000\,dt = 750{,}000\ €.$$

For a bond the mean expected exposure seems to be a reasonable concept because, if the bond defaults in more than 6 months, a much larger amount of money is expected to be lost than its current exposure.

(10) Maximum exposure

Similar to the mean expected exposure is the maximum exposure. It is defined as

$$MAX = \max_{t \le T} \left(\mathbf{E}(V(t)), 0 \right).$$

It can well be applied for fixed income products, especially for limitation if a bank wants to take account of peak exposures and stabilize the employment of credit limits.

(11) Artificial spread curves

For most loans and bonds on the banking books no market prices exist because the secondary markets for these instruments are highly illiquid as a result of default risk problems which cause market intransparency. For this reason, usually the trades' book value is employed for credit risk management purposes. This leads to valuation inconsistencies, though, because an instrument's book value typically deviates from its present or market value. However, if risk-free interest rates and the client's default probabilities are known, risk adjusted spread curves can be derived and the loan's exposure can be calculated.

Let us take a 2-year loan as an example with a book value or notional of K = 100,000 € and a fixed interest of r = 10% p.a.. Interest is paid once a year. The client's first year default probability has been assessed as p_1 = 0.6% and his second year's was found equal to p_2 = 0.4%. His recovery rate, in case of default, is assumed to be fixed at RR = 50%. The risk-free interest rate is r_1 = 4.5% for one and r_2 = 5% for two years.

We begin with the calculation of a 2-year risky spread curve. A zero bond that pays out 1 € in one year and that was closed with the client is worth

$$e^{-r_1} \cdot \left[\underbrace{1-p_1}_{no-default} + \underbrace{p_1 \cdot RR}_{default} \right] = e^{-r_1 - s_1}$$

which implicitly defines the 1-year spread s_1. Thus, we have

$$s_1 = -\ln\left[\underbrace{1-p_1}_{no-default} + \underbrace{p_1 \cdot RR}_{default} \right] = 0.3\%$$

Analogously the 2-year spread s_2 is

$$s_2 = -\frac{1}{2}\ln\left[\underbrace{(1-p_1)(1-p_2)}_{no-default} + \underbrace{(1-(1-p_1)(1-p_2)) \cdot RR}_{default} \right] = 0.25\%\,.$$

We can now value the loan

$$V = r \cdot K \cdot e^{-r_1 - s_1} + (1+r) \cdot K e^{-(r_2+s_2)\cdot 2} = 108{,}566.95\,€.$$

c) Overview over applications

To sum up what was said about the applications of exposure concepts, we give an overview in Table 6.

	Portfolio management	**Limitation**	**Equity allocation**
Present value	O	O	O
Current exposure	+	O	+
Peak exposure	-	+	-
Portfolio potential exposure	++	++	++
Mean expected exposure	++	++	++
Maximum exposure	-	+	-

Table 6: Typical applications of exposure concepts

D. Loss and Recovery Rates

One of the major factors of uncertainty in credit risk management is the loss of exposure conditional to default or, conversely, the recovery conditional to default. It is clear that the credit risk of a contract would be zero even if its default probability is high if its loss conditional to

default were zero. On the other hand, its credit risk would reach its maximum if its loss rate were 100%.

In the following, we will refer to the proportion of the exposure that is lost conditional to default as loss rate λ, and on the respective recovered proportion as recovery rate r. Both quantities are related by the equation

$$r = 1 - \lambda.$$

1. Influence factors

Empirical investigations[186] have shown that loss and recovery rates are particularly dependant upon four factors:

1. *The concept of default.* If the definition of default[187] chosen is strict, i.e. if it is an absorbing state that implies the insolvency of the counterparty or counterparties owning a contract, recovery rates are lower on average than in case of a weak comprehension of default where the state is not absorbing. In this second situation many contracts formerly in default are finally fulfilled and not liquidated prematurely meaning that no loss occurs at all.

2. *The seniority ranking of debt.* All kinds of transactions do not have the same priority or seniority if the assets of a counterparty in insolvency are liquidated and the proceeds are used to repay the obligations to the counterparty's creditors. On the contrary, there is a well defined hierarchy of privileges[188] that regulates the sequence in which claims are satisfied in a liquidation process. The higher a contract's seniority ranking is, the higher *ceteris paribus* is the recovery. Table 7 gives an impression of the effect of a transaction's seniority on recovery rates.

Seniority	**Mean Recovery** (% of face value)
Senior secured	63.45%
Senior unsecured	47.54%
Senior subordinated	38.28%
Subordinated	28.29%
Junior subordinated	14.66%

Table 7: Mean default recovery rates on public bonds by seniority[189]

[186] Cf. e.g. to Altman et al. (1996), Standard & Poor's (1992), and Moody's (1997). All those studies put the focus on US or multinational companies. We are not aware of a similar study for European companies or for private customers.

[187] See above section Definitions of default on page 16.

[188] Cf. to Caouette et al. (1998), p. 206, for details.

[189] See Moody's (1997), exhibit 5.

3. *Instrument type*. The impact of the instrument type on recoveries is a consequence of the instruments' difference in seniority. Usually, derivative transactions as a whole have the most junior ranking being only senior to equity. Thus, a company's shareholders only receive a recovery if all other of the company's creditor's could be paid. The argument also shows that results as quoted in Table 7 for public bonds can not be transferred to other product types.

4. *Industry of the obligor*. A company's asset structure and especially the assets' liquidation value strongly depend upon the industrial sector an obligor belongs to. Caouette et al.[190] quote a working paper by Altman and Kishore in which it is found that in the US the chemical industry (63%) and the public utilities sector (70%) have the highest average recovery rates while most other industries range around 30-40%. The lodging, hospital and nursing facilities only have a mean observed recovery rate of 26%.

2. Random Recoveries

Rating agencies usually not only publish mean recovery rates for different seniority classes of bonds, but also their standard deviation because the uncertainty in recovery rates observed in different cases is considerable even within the same seniority ranking. However, the knowledge of the recovery rates' standard deviation renders it possible to take account of this uncertainty and to understand them as a random variable. For convenience rather than for empirical reasons, the beta distribution is frequently chosen as a model for recovery rates since it can be fitted by its mean and standard deviation, only takes values between zero and one, and can easily be simulated by inversion of the cumulative distribution function.

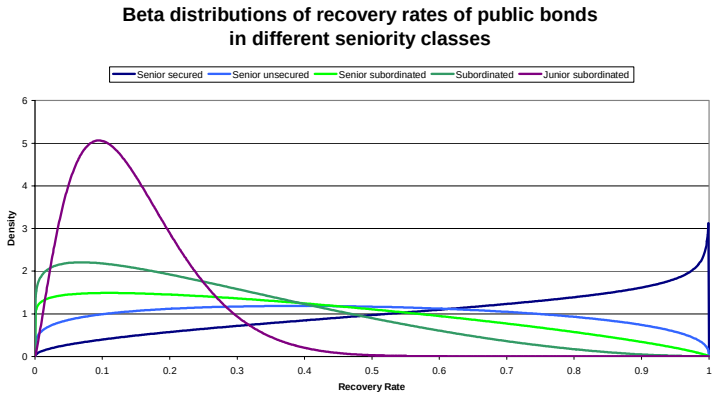


Figure 29: Beta distributions of recovery rates of public bonds in different seniority classes[191]

[190] Caouette et al. (1998), p.211.

[191] See Moody's (1997), exhibit 5.

Figure 29 gives an impression of the shape of the densities of the resulting beta distributions of recovery rates if they are fitted with the estimated moments of the distribution[192].

Since the seniority ranking is a feature of the single contract, the corresponding recovery rates cannot be directly used if all transactions belonging to a specific client are considered as one exposure and a subportfolio that defaults simultaneously. Therefore, an aggregation rule is required in this situation:

Let $X_1,\ldots, X_n$ be the exposures of the single transactions in the subportfolio that is owned by the counterparty under consideration with expected recovery μ_i and standard deviation σ_i for $i = 1,\ldots,n$. Let

$$\alpha_i = \frac{X_i}{\sum_{j=1}^{n} X_j}$$

for $i = 1,\ldots,n$, be the share of the ith transaction in the exposure of the subportfolio. Then the mean recovery μ of the subportfolio can be calculated as

$$\mu = \sum_{i=1}^{n} \alpha_i \cdot \mu_i$$

and the standard deviation σ as

$$\sigma = \sum_{i=1}^{n} \alpha_i \cdot \sigma_i$$

since the recoveries can be assumed to be fully dependent because they refer to the same counterparty.

Note that the resulting distribution would not necessarily be a beta distribution again if the distributions of the single recovery rates are beta. This is not a serious problem, though, since the assumption of a beta distribution was arbitrary. Therefore, it is conceptually no difference if only the resulting distribution is assumed to be beta.

---

[192] The beta distribution with parameters a and b has the density function

$$f_{a,b}(x) = \frac{\Gamma(a+b)}{\Gamma(a)\cdot\Gamma(b)} x^{a-1}(1-x)^{b-1}.$$

where $\Gamma(\cdot)$ is the gamma function. The parameters are related to the mean μ and the standard deviation σ by

$$a = \frac{\mu^2(1-\mu)}{\sigma^2} - \mu \quad \text{and} \quad b = \frac{\mu(1-\mu)^2}{\sigma^2} - (1-\mu).$$

3. Practical problems

The valuation of assets after a default is not straight forward since for most 'used' assets the market is highly illiquid so that just book values, but no market prices exist. Moreover, a full settlement after an insolvency usually takes years so that definite results about losses and recoveries cannot be obtained quickly. A practical compromise has turned out to be relatively accurate is to estimate the value of recoveries one month after default[193].

Finally, the legal environment can have a major impact on recovery rates. This implies that the results of recovery studies for the US cannot easily be transferred to European situations and underlines the need for investigations for countries outside the US.

E. Pricing

An important component in the management of a single client's credit risk is the correct and risk adequate pricing of new transactions. The pricing ensures that the client himself pays for the expected cost that the financial institution has to carry when the transaction is agreed upon. This includes the refinancing costs and the cost of risk, i.e. the expected loss of the trade and the costs of provisions for the unexpected loss.

As examples for the general methodology, we will derive pricing formulas for a fixed rate bond as an interest rate product and for a European equity call option. For simplicity, we will ignore the possibility of rating transitions and only consider the two states of default and no default as influences of the client's credit risk on the value of the product.

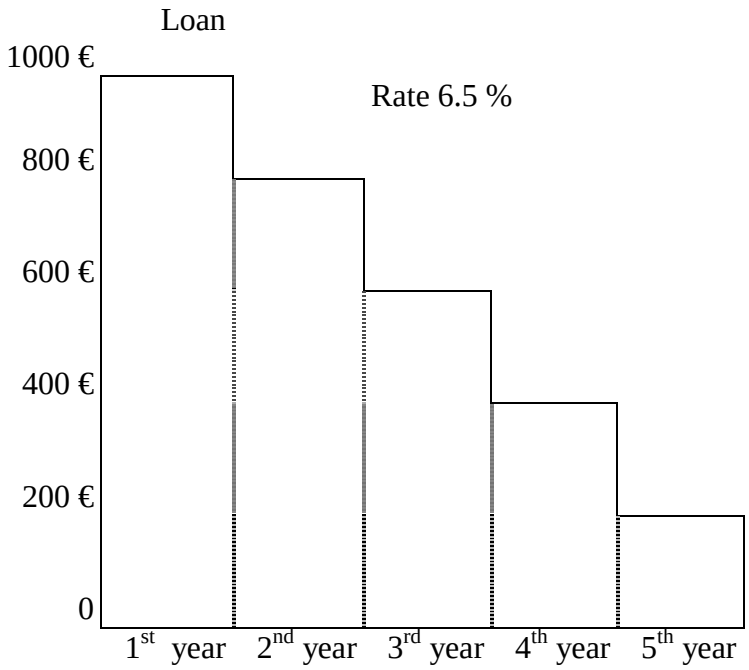
1. Pricing of a fixed rate bond

A fixed rate bond is a loan over a certain number of periods of time at a fixed rate. At the end of each period the accrued interest and an amortization are paid[194].

As an example, we take a loan of 1,000 € over five years where 5 annuities are paid at a rate of 6.5% plus an amortization of 200 € at the end of each year. The development of the notional of this bond has the following structure:

[193] Cf. Gupton et al. (1997), p. 77.

[194] Note that the amortization can be positive or negative. If it is negative, we also speak of a step-up bond.

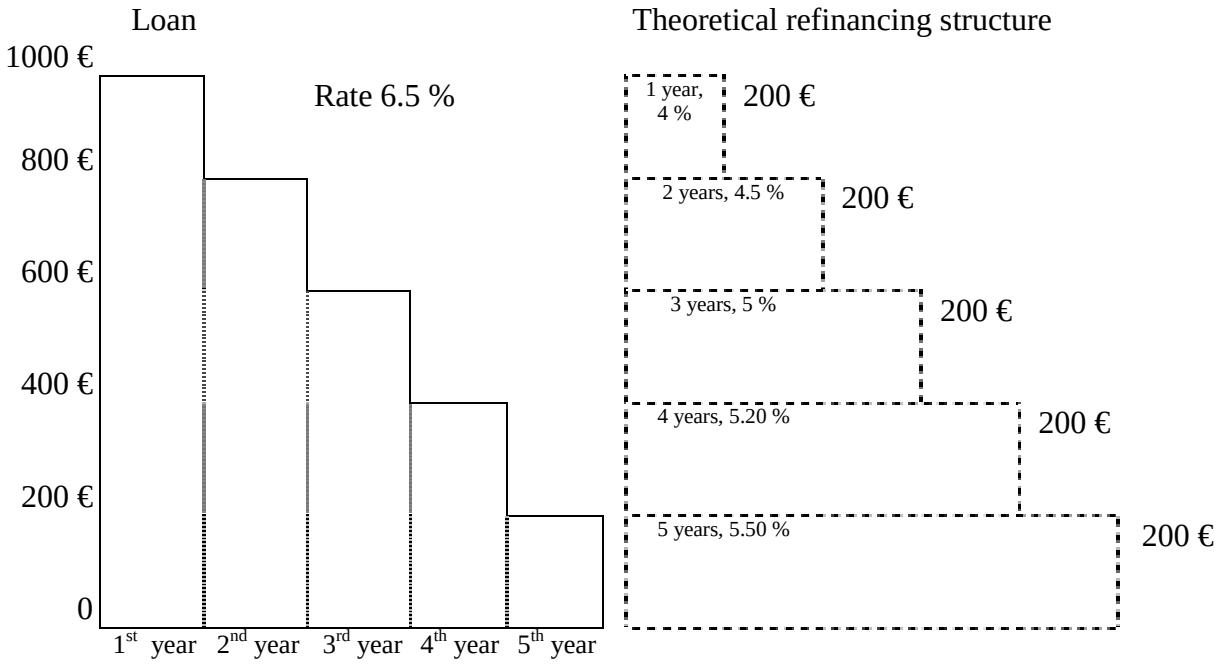


We define the "fair" price of a transaction as the price where expected revenues equal expected costs. We, thus, look for the interest rate *r* where

$$\text{revenues}(r) = \text{costs}(r).$$

The revenue side consists of two components: interest payments and amortizations as long as the bond is served as intended, and the recovery in case of default. Similarly, the cost side comprises the components refinancing expenses and cost of equity.

To exclude costs or revenues from term transformation in the above example, we suppose a refinancing structure that is identical to the amortization structure. Additionally, we assume interest rates at which the bank itself can lend money:



For the ease of presentation, we fix the following notation:

- Let p_i be the probability that the client defaults in year i for $i = 1,\ldots,n$, given that he has not defaulted before. For the example we assume: $p_1 = 1\%$, $p_2 = 1.5\%$, $p_3 = 1.2\%$, $p_4 = 1.8\%$, $p_5 = 1\%$.
- Let r_i^f be the bank's reference interest rate for a straight bond with annual coupon payments with a maturity at the end of year i for $i = 1,\ldots,n$. In our example, we have: $r_1^f = 4\%$, $r_2^f = 4.5\%$, $r_3^f = 5\%$, $r_4^f = 5.2\%$, $r_5^f = 5.5\%$.
- Let r_i^* be the risk free interest rate of a zero bond with maturity at the end of year i for $i = 1,\ldots,n$. For simplicity we assume $r_i^* = r_i^f$ for $i = 1,\ldots,n$, in the example.
- Let c_i be the bond's notional capital in year i for $i = 1,\ldots,n$. In the example, we have $c_1 = 1000$€, $c_2 = 800$ €, …, $c_5 = 200$ €.
- Let a_i be the amortization at the end of year i for $i = 1,\ldots,n$. $a_i = 200$ € for all i in the example.
- Let RR be the client's expected recovery rate.

- Let q be the fraction of the respective notional[195] that has to be ensured by equity. We assume that q is constant over time. A supposition that can easily be relaxed. For the moment, we presume q as exogenously given.
- Let r^e be the return on equity intended by the bank. We assume that the bank invests its equity in risk-free long term tradable bonds where it receives an annual interest rate of $r^* > r_i^f$ for all i to reduce opportunity costs. For the example, we choose r^e = 15% and r^* = 8%.
- We assume that the counterparty can only default right before an interest payment because in this situation the default incurs the maximum loss due to the accrued interest.

The following theorem then defines the fair risk premium:

Theorem 1:
With the above notation the fair price of a fixed rate bond is given as the solution r of the equation

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{i}(1-p_j)\frac{r\cdot c_i+a_i}{(1+r_i^*)^i}+\prod_{j=1}^{i-1}(1-p_j)p_i\cdot RR\cdot c_i\frac{1+r}{(1+r_i^*)^i}\right)$$
$$-\sum_{i=1}^{n}\left(a_i\left(q\cdot\left(r^e-r^*\right)\right)\cdot\sum_{j=1}^{i}\frac{\prod_{k=1}^{j-1}(1-p_k)}{(1+r_j^*)^j}+\sum_{j=1}^{i}\frac{r_i^f}{(1+r_j^*)^j}+\frac{a_i}{(1+r_i^*)^i}\right)=0$$

Proof:

Let's look at the income side first:

At the end of year i two kinds of cash flows are expected: an interest plus an amortization payment. The probability that the cash flows are carried out as intended is equal to the probability that the client has not defaulted yet, thus, $\prod_{j=1}^{i}(1-p_j)$. Hence, the expected value of the cash flows in year i is

$$\prod_{j=1}^{i}(1-p_j)(r\cdot c_i+a_i).$$

This amount has a present value of

[195] According to Grundsatz 1 § 6.1.1. the amount of equity to be allocated to compensate for the bond's default risk is a fraction of its notional. Alternatively, one could supply equity as a fraction of the bonds present value. In this case, a similar formula as in Theorem 1 would result.

$$\prod_{j=1}^{i}\left(1-p_j\right)\frac{r\cdot c_i+a_i}{\left(1+r_i^*\right)^i}.$$

If the counterparty defaults in year *i*, the exposure $c_i(1+r)$ is put into question. According to our assumptions a present value of

$$RR\cdot c_i\frac{1+r}{\left(1+r_i^*\right)^i}$$

can be recovered. The probability that the client defaults in year *i* equals $\prod_{j=1}^{i-1}\left(1-p_j\right)p_i$.

We, therefore, have a total expected revenue of

$$\sum_{i=1}^{n}\left(\prod_{j=1}^{i}\left(1-p_j\right)\frac{r\cdot c_i+a_i}{\left(1+r_i^*\right)^i}+\prod_{j=1}^{i-1}\left(1-p_j\right)p_i\cdot RR\cdot c_i\frac{1+r}{\left(1+r_i^*\right)^i}\right).$$

The cost function can be developed analogously:

We look at each part of the refinancing structure separately. For all *i*, the part with maturity at the end of year *i* has the same notional as the respective amortization. The whole amount that is handed out to the client is refinanced over the capital market. However, a fraction *q* of the notional has to be provided in form of equity to compensate for the client's default risk if no default has occurred yet. Thus, this fraction has to earn the excess return on equity or the risk premium the bank intends. Hence, each year for year 1 to *i* nominal interest payments of $a_i\left(\prod_{k=1}^{j-1}\left(1-p_k\right)\cdot q\cdot\left(r^e-r^*\right)+r_i^f\right)$ are due. Discounted and aggregated over the years we have

$$a_i\left(q\cdot\left(r^e-r^*\right)\right)\cdot\sum_{j=1}^{i}\frac{\prod_{k=1}^{j-1}\left(1-p_k\right)}{\left(1+r_j^*\right)^j}+\sum_{j=1}^{i}\frac{r_i^f}{\left(1+r_j^*\right)^j}.$$

Finally, the amortization has to be paid back each year, $\frac{a_i}{\left(1+r_i^*\right)^i}$. Altogether the cost function is given by

$$\sum_{i=1}^{n}\left(a_i\left(q\cdot\left(r^e-r^*\right)\right)\cdot\sum_{j=1}^{i}\frac{\prod_{k=1}^{j-1}\left(1-p_k\right)}{\left(1+r_j^*\right)^j}+\sum_{j=1}^{i}\frac{r_i^f}{\left(1+r_j^*\right)^j}+\frac{a_i}{\left(1+r_i^*\right)^i}\right).$$

This completes the proof.

Note that we did not need the assumption that markets are efficient or that there exist no arbitrage possibilities. Conversely, the result can be used to find arbitrage opportunities and to avoid arbitrage against the bank and, thus, to make markets more efficient.

In the example, the risk-free fair price is 5.07%. In this case, no equity is necessary to finance the deal.

Under risk and different values of the recovery rate and the necessary amount of equity, further results are:

	RR = 90%	**RR = 60%**	**RR = 30%**	**RR = 0**
q = 3%	5,40%	5,81%	6,21%	6,62%
q = 5%	5,54%	5,95%	6,35%	6,77%
q = 8%	5,75%	6,16%	6,57%	6,98%
q = 11%	5,96%	6,37%	6,78%	7,19%

Table 8: Fair prices of defaultable fixed rate bonds

Equivalently to the fair price of a defaultable bond, we can state the fair spread of the bond as the excess of the fair rate over the risk-free rate:

	RR = 90%	**RR = 60%**	**RR = 30%**	**RR = 0**
q = 3%	0,33%	0,74%	1,14%	1,55%
q = 5%	0,47%	0,88%	1,28%	1,70%
q = 8%	0,68%	1,09%	1,50%	1,91%
q = 11%	0,89%	1,30%	1,71%	2,12%

Table 9: Fair spreads of defaultable fixed rate bonds

It is evident that large amounts of equity necessary to compensate for the risk due to regulatory requirements, poor diversification of the bond in the portfolio (large values of q) and a low seniority (small values of RR) can cause the price of the bond to rise sharply. This picture appears to be even more drastic if we look at the net commercial margin of the bond, i.e. the excess of the supposed traded rate of 6.5% over the fair rate:

	RR = 90%	**RR = 60%**	**RR = 30%**	**RR = 0**
q = 3%	1,10%	0,69%	0,29%	-0,12%
q = 5%	0,96%	0,55%	0,15%	-0,27%
q = 8%	0,75%	0,34%	-0,07%	-0,48%
q = 11%	0,54%	0,13%	-0,28%	-0,69%

Table 10: Commercial margins of defaultable fixed rate bonds

If expected recoveries are low and / or capital requirements are high, the commercial margin quickly becomes negative, albeit the relatively high spread of the traded rate over the risk-free fair rate of 143 basis points.

Note that by defining

- the difference between the traded rate and the risk-free rate as the "gross commercial margin",
- the difference between the traded rate and the risk carrying rate as the "net commercial margin",
- and the difference between the risk carrying fair rate and the risk-free rate as the "fair risk spread"

the results can be integrated into the standard concept of funds transfer pricing.

2. Pricing of a European option

As a second example for the calculation of risk premiums, we consider a European stock call option. We assume that the direct counterparty of the option transaction and the issuer of the underlying stock are not the same. We further suppose that the stock issuing company cannot default so that standard techniques can be used to price the option in a default-risk-free environment[196]. Finally, we assume that the development of the underlying stock's market price and the default behavior of the direct counterparty are stochastically independent.

Additional to the notation above, we define the following abbreviations:

- Let c be the default-free value of the option when refinance costs are neglected. For instance, c could be the result of a Black-Scholes analysis.
- Let T be the maturity of the option.
- Let C be the option's market value and A be an add-on on the market value so that the option's potential exposure[197] is given by $C + A$.
- Let r_T^z be the risk-free interest rate and r_T^f the bank's refinance rate corresponding to the maturity of the option.
- Let p be the counterparty's probability of default.

If we assume the option's market value C to equal c_d, it is straight forward to see that the fair price c_d of the defaultable stock option is then given by

[196] If the underlying stock could default, the distributional assumptions that lead, for instance, to the Black-Scholes formula would be inadequate.

[197] According to Grundsatz 1 § 10 the amount of equity to be allocated to compensate for the option's default risk is a fraction of its potential exposure, thus, a fraction of its market value plus an add-on to take account of a potential increase in market value.

$$(1-p+p\cdot RR)\cdot c-\frac{\left(\left(\left(1+r^{e}\right)^{T}-\left(1+\max\left(r^{*},r_{T}^{f}\right)\right)^{T}\right)\cdot q+\left(1+r_{T}^{f}\right)^{T}\right)\cdot c_{d}+\left(\left(1+r^{e}\right)^{T}-\left(1+r^{*}\right)^{T}\right)\cdot q\cdot A}{\left(1+r_{T}^{z}\right)^{T}}=0\cdot$$

This formula can be solved for c_d as

$$c_{d}=\frac{(1-p+p\cdot RR)\cdot\left(1+r_{T}^{z}\right)^{T}\cdot c-\left(\left(1+r^{e}\right)^{T}-\left(1+r^{*}\right)^{T}\right)\cdot q\cdot A}{\left(\left(1+r^{e}\right)^{T}-\left(1+\max\left(r^{*},r_{T}^{f}\right)\right)^{T}\right)\cdot q+\left(1+r_{T}^{f}\right)^{T}}.$$

Note that for positive $p > 0$ or $q > 0$ or $r_T^f > 0$, c_d is strictly smaller than c.

If the issuer of the stock cannot default, the same formula can be used for European puts. For American puts and calls, the formula understates the value of the defaultable option since it can be exercised early implying that the default probability of the direct counterparty could be smaller than p.

3. Equity allocation

So far in this chapter, we have assumed that the amount of equity the bank has to supply to compensate for a contracts default risk is given exogenously.

This is certainly true if the regulatory requirements for equity allocation are used to calculate risk premiums. In this case, equity is a specific fraction of a quantity that depends on the instrument and the type of its underlying asset if the instrument is a derivative. Both, fraction and base are defined in the Capital Adequacy Directive (CAD) of the European Union and have been incorporated into national law such as the Grundsatz 1 for Germany.

An alternative approach to determine equity allocation is to use the results of the value at risk calculation from portfolio analysis. Here, the portfolio value at risk to a specified confidence level is the total amount of equity necessary for the full portfolio. The fraction q_i of transaction i's exposure ($i = 1,\ldots,n$) that has to be covered by equity can then be calculated as the transaction's marginal value at risk m_i relative to the sum of all marginal values at risk times the portfolio value at risk m relative to the transaction's exposure E_i, thus,

$$q_{i}=\frac{m_{i}}{\sum_{j=1}^{n}m_{j}}\cdot\frac{m}{E_{i}}.$$

Or equivalently a transaction's economic capital is given as

$$q_{i}E_{i}=\frac{m_{i}}{\sum_{j=1}^{n}m_{j}}\cdot m$$

Note that if exposures are small, a client's marginal value at risk is approximately linear in his exposure. It is, therefore, in most cases not necessary to calculate marginal values at risk at the transaction level, but it is sufficient to consider marginal values at risk of clients.

Both methodologies, the portfolio and the regulatory approach, are not equivalent because the method chosen by the CAD neglects correlations among clients, recovery rates, and the degree of concentration of a client within the portfolio.

As an example, we choose a homogenous portfolio of identical clients where each client's exposure is negligible relative to portfolio value. The contract to price is a one period zero bond. The bank's reference rate and the risk-free discount rate are set as $r^f = r^z = 4\%$.

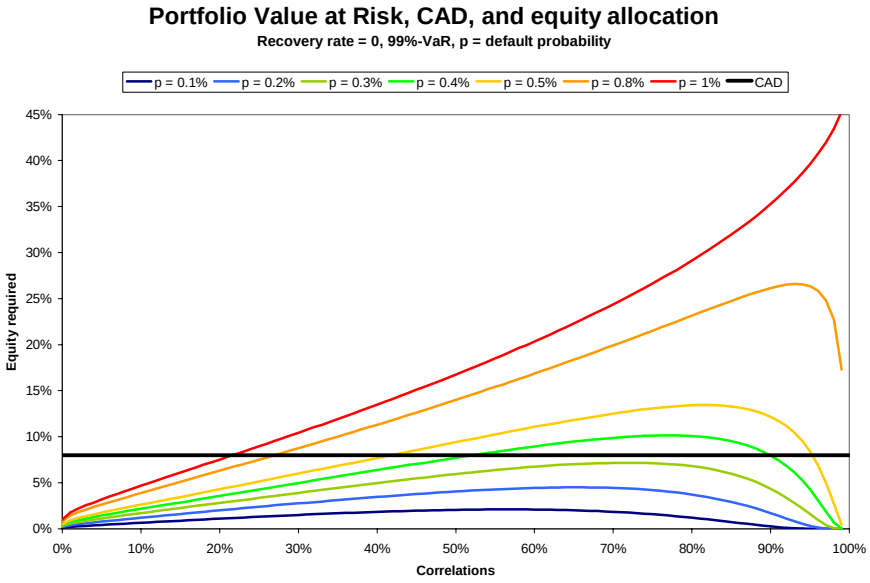


Figure 30: Portfolio analysis, CAD, and equity allocation

Figure 30 shows the capital requirements that stem from the 99%-VaR in a homogenous portfolio if recoveries are zero. It is obvious that for low probabilities of default the CAD overstates the capital requirements to compensate for the 99%-VaR, at least for this highly diversified type of portfolio. However, it understates capital requirements if correlations and default probabilities increase implying that speculative grade portfolios tend to be underfunded by the CAD.

This observation carries over to risk spreads. As stated in Theorem 1, risk premiums depend, among other things, on the client's default probability and on the amount of equity necessary to cover the contract's default risk. Since banks require a risk premium on the equity they allocate on top of the market price of a risk-free investment. Thus, risk spreads increase with equity requirements.

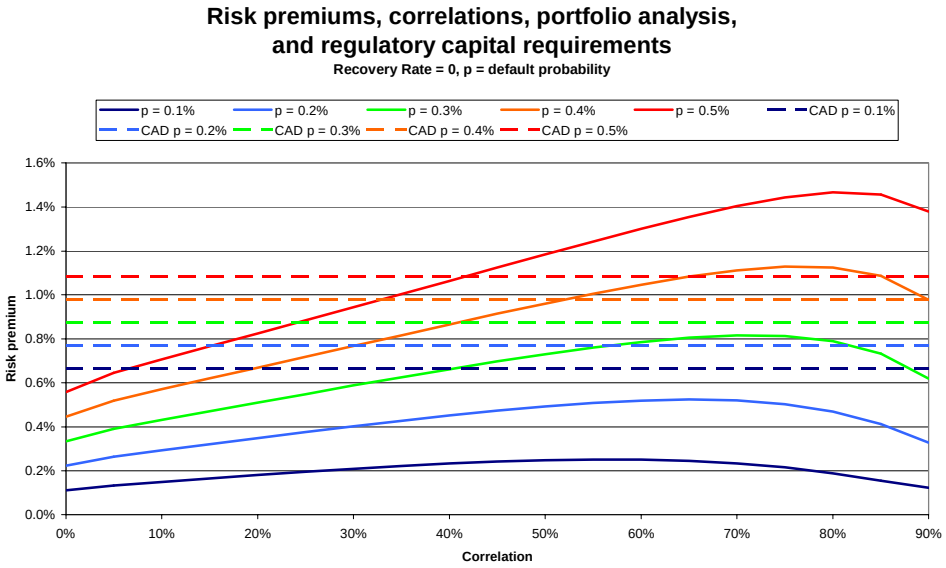


Figure 31: Risk premiums, correlations, portfolio analysis, and regulatory capital requirements

Figure 31 shows that bonds in high quality portfolios are overpriced if a bank uses the CAD for capital allocation. Conversely, for low quality portfolios where default probabilities and / or correlations are high risk premiums are too low.

II. The credit risk of multiple clients

Portfolio risk takes into account the interaction of individual exposures. In the following, we present various ways to model dependencies among clients. We generalize the normal correlation model which is used in Credit Metrics and the Vasicek-Kealhofer model and show that it represents exactly the risk minimal case in the generalized model. Subsequent sections discuss the concepts of dependence used in Credit Risk+ and Credit Portfolio View. In particular, it is shown that the aggregation of risks may lead to a systematic overestimation of portfolio risk in heterogeneous portfolios in Credit Risk+. Finally, we suggest the consideration of event driven dependencies in the CRE model.

The description of dependencies completes the model specific analysis and the modeling process. The next chapters present model-independent estimation techniques for the portfolio loss distribution and relevant risk measures. Moreover, we identify the most important characteristics of the estimators, and provide confidence intervals for the results. As the main result of this section, we show that a systematic error occurs in the value at risk and shortfall estimation if the number of simulations runs is not sufficiently large with regard to the required confidence level.

We conclude with the description of risk analysis and risk management techniques. As an illustration of their application, a risk management cockpit is constructed, an analysis is performed for an example portfolio, and the subject of credit derivatives is briefly touched. As a complementary approach to improve portfolio quality that does not depend upon the present portfolio composition, we develop an algorithmic and fully automizable method that minimizes portfolio shortfall under side-constraints and again perform an example portfolio optimization.

A. Concepts of dependence

A crucial step in the construction of a credit portfolio model is the description of dependencies between clients[198]. This stage in the modeling process of the portfolio structures is essential as there is striking empirical evidence that the default behavior of counterparties in real world portfolios is not independent.

[198] If exposures are not tied to single clients, one can also model dependencies between exposures. See also section I.B.

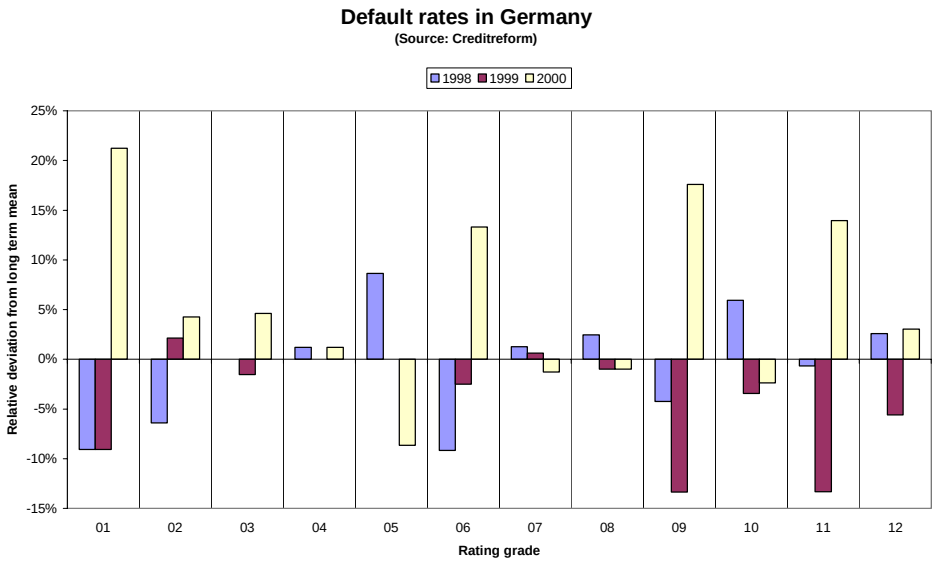


Figure 32: Default rates of corporates in Germany for different rating grades

This can easily be seen because if defaults were independent observed default rates in large portfolios could be expected to be identical to their long term mean due to the law of large numbers. It is obvious from Figure 32 that this is not true. Figure 32 is based on the analysis of 2.5 million corporates in Germany with rating grades comprising between 12.000 (grade 01) and 538.000 (grade 05) companies.

Besides the clients' individual characteristics such as probability of default, exposure, loss given default etc. the modeling of interrelations between clients decisively determines the resulting portfolio loss distribution and the risk found in the portfolio.

Following the suggestions in the literature, we would like to discuss the following general concepts of dependence:

- The normal and the generalized correlation model (cf. the Vasicek-Kealhofer model by KMV, Credit Metrics, and the CRE model).
- Stochastic default probabilities (cf. Credit Portfolio View, Credit Risk +)
- Dependencies due to common exposure to country risk (cf. the CRE model, Credit Portfolio View)
- Dependencies due to a special relationship between individual clients (cf. the CRE model)

1. The normal correlation model

a) The Vasicek-Kealhofer model

The normal correlation model was the first fully worked out model of dependencies between clients in a credit portfolio. It was mainly developed by KMV Corporation[199] in the mid 1990's.

Based on Robert Merton's seminal article on corporate default risk from 1974[200], it is assumed that a company defaults if its firm value falls below the face amount of its debt at the time the debt is due[201] because its proprietors are better off if they hand the firm over to the creditors instead of repaying the debt.

Moreover, in the Merton-model, the firm value follows a geometric Brownian motion. This supposition allows for a straight forward generalization of Merton's single firm model to a portfolio model by assuming the joint firm value processes of a set of n firms to follow a multivariate geometric Brownian motion with pairwise linear correlations ρ_{ij} between firm i and firm j, $i, j = 1,\ldots, n$, of the logarithmic increments. Still every single firm defaults if its asset value is inferior to its liabilities at the debt's maturity, however, default events can now be dependent due to the correlated asset values.

As a simplification, KMV supposes that all firms' debts have a standardized time to maturity of one year. Since the logarithm of the firm value process, i.e. the asset return process, at time $t = 1$ is normally distributed with given mean and variance[202] and since the linear correlation ρ is invariant under linear transformations[203], we can presume without loss of generality that the firms' joint asset return process V_1 is normally distributed with mean

$$\text{Mean}(V_1) = 0$$

and variance

$$\text{Variance}(V1) = \text{Corr}(V1) = \rho = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \rho_{n-1,n} \\ \rho_{n1} & \cdots & \rho_{n,n-1} & 1 \end{pmatrix}.$$

199 See Kealhofer 1993.

200 See also sections I.C.1. and I.C.2. above.

201 It is also supposed that all of the firm's liabilities consist of a zero bonds of the same seniority due at the same point in time. For a full list of assumptions of Merton's model confer to section I.B.1.

202 See section I.B.1. above.

203 Note that this implies that asset return correlations are time invariant. The time horizon t only enters into the calculation of portfolio risk through firms' default probabilities.

Once default probabilities p_i, $i = 1,\ldots, n$, have been calculated for the single firms with the original Merton approach, the firms' debts can be replaced by abstract default thresholds d_i given by

$$d_i = \Phi^{-1}(p_i)$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution (see Figure 33).

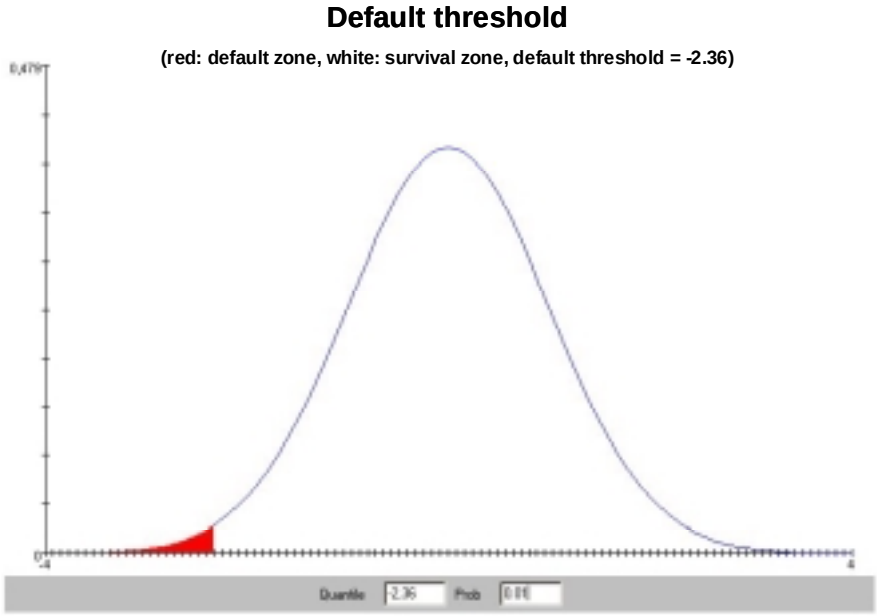


Figure 33: Abstract default threshold for a firm with annual probability of default of 1%

Now firms' joint default behavior can be simulated by drawing random numbers from the multivariate standard normal distribution with the specified correlation matrix.

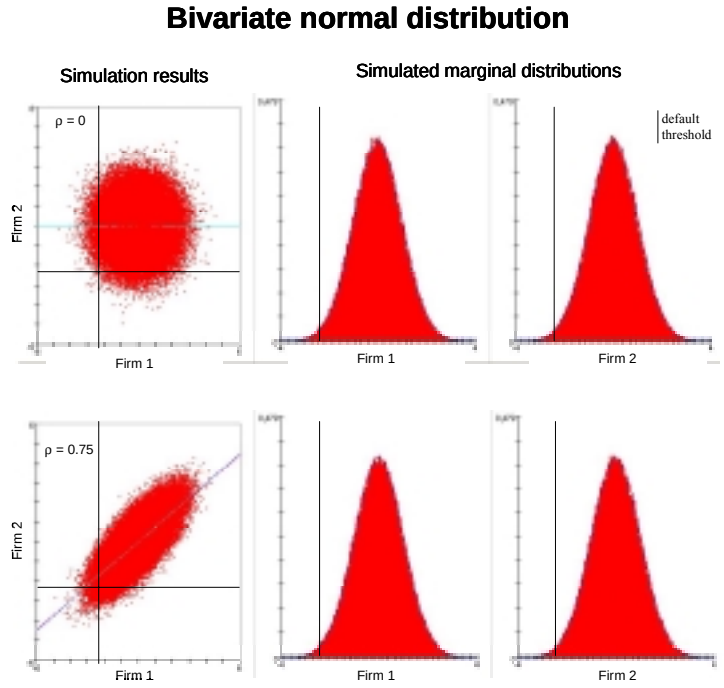


Figure 34: Simulated bivariate normal distributions and marginals

Figure 34 illustrates that firms' marginal[204] asset return distributions always remain unchanged independent of their correlation. Their joint distribution, however, is distinctly shaped by the correlation coefficient ρ. Note that firms' joint default probability, i.e. the probability that both firms' sampled asset returns are inferior to the respective default thresholds, increases sharply with their correlation.

The remaining task is to estimate the asset return correlation matrix. Theoretically, the estimation can be performed based on historical data on firms' asset value processes using standard techniques from time-series analysis. In practice, this is much more difficult, though, because realistic portfolios contain contracts with thousands of clients while the number of correlation coefficients to be estimated increases quadratically with the number of clients[205]. For this reason, KMV defines characteristic systematic risk factors for countries and industry sectors. Each individual firm's asset return process is regressed against the factor return processes. Asset return correlations result in turn from regression parameters and factor return correlations[206].

[204] Note that the marginal distributions do not represent firms' exact asset value distributions at time $t = 1$, but rather an analogous and stochastically equivalent distribution. While keeping the concept of linear correlations between marginal distributions, later generalizations of the KMV-model, such as Credit Metrics or the CRE model, intentionally move away from the concrete interpretation of the distributions as asset value processes.

[205] Having n clients, $n/2 \cdot (n-1)$ correlation coefficients are to be estimated.

[206] Confer to Lipponer 2000, p. 48f., for details.

b) Credit Metrics

A variation of the KMV approach was suggested by Gupton, Finger, and Bhatia in the model known as Credit Metrics. In Credit Metrics the Vasicek-Kealhofer portfolio model is separated from Merton's option pricing method to calculate default probabilities. Default and also transition probabilities from one grade to the other are considered as being exogenously given through company ratings so that the firms' asset value processes are no longer relevant to fit the model. This enlarges the applicability of the model from public to all externally or internally rated companies.

The concept of dependence between counterparties in Credit Metrics is identical to the KMV approach. The interpretation of the multivariate normal and the marginal distributions as firms' joint and marginal asset return distributions is now purely intuitive in the Credit Metrics context, so that it would be more correct to rather speak of abstract risk index distributions. Their main role in the model is to extend the individual firms' transition probabilities as given by their ratings to joint transition probabilities for the entire portfolio of firms.

All other changes in Credit Metrics with regard to the KMV portfolio model are technical in nature and motivated by the lack of actual asset value data. This concerns especially the calculation of correlations between companies' risk indices since the regression of asset value processes against systematic factor processes is no longer possible. For public companies Gupton et al. propose the consideration of equity prices as a proxy for asset values and to use them to define systematic risk factors for industry sectors and countries[207].

Individual firms are mapped to systematic risk factors by their representation in the respective sectors and countries. Gupton et al. do not tackle the question how a company's representation in a certain environment can be measured. However, this gap can be filled by using a firm's turnover or profit in a certain market relative to its total turnover or profit as an indicator for its representation in that market.

To avoid the problem that two identical firms are entirely correlated so that they either both default at once or do not default at all, Gupton et al. introduce the concept of idiosyncratic risk. A company's idiosyncratic risk, as being opposed to its systematic risk, signifies the percentage of the variance of its risk index that cannot be explained by systematic factors. Gupton et al. do not state how systematic and idiosyncratic risk components can be estimated. For public companies, however, systematic risk is given as the R^2 of the regression of the firms

[207] Gupton et al. 1997, p. 92ff.

equity returns against the returns of its systematic risk factor. Idiosyncratic risk now can be calculated as 1 - R^2.[208]

It is a slight imperfection in Credit Metrics that Gupton et al. extend on the one hand the Vasicek-Kealhofer model to non-public companies by the use of rating data for default probabilities, but on the other hand do not provide a means to estimate the risk index correlations of this type of company. In this version, Credit Metrics does effectively not go beyond the Vasicek-Kealhofer model.

This problem can be solved if systematic risk factors are chosen that do not involve equity prices. We suggest the gross value added as a suitable indicator for systematic risk since it has a number of major advantages:

- It has been calculated by statistical bureaus in all economically important countries for all relevant industry sectors for many years so that relatively long time series are publicly available.
- It can be calculated for individual companies by subtraction of purchases from turnover.
- It is a more basic and stable indicator for economic performance than equity price indices because it is not superimposed by speculative influences.

c) Homogenous Portfolios

For general portfolios the normal correlation model can only be solved to get the portfolio loss distribution using simulation techniques. For the special case of infinitely large homogenous portfolios an analytic solution exists and the loss distribution is given as a parametric function.

A homogenous portfolio is defined as consisting only of identical clients in terms of probabilities of default *p*, exposure *E*, risk index correlations[209] ρ, and expected loss given default

[208] Note that Gupton et al. define a firm's systematic risk as the part of the standard deviation of its risk index that is explained by systematic risk factors (p. 100). Though being equivalent to the definition we stated above – it is just the square root of our definition -, it has the drawback that in this setting systematic and idiosyncratic risk usually sum to more than 100%.

[209] Risk index correlations ρ have to be non-negative because otherwise the sum of risk indices would have a negative variance *v* since *v* is given as

$$v = n + \rho \cdot n / 2 \cdot (n - 1)$$

where *n* is the number of clients in the portfolio. Thus, for negative ρ the maximum number of clients in a homogenous portfolio is

λ[210]. Without loss of generality, we assume that all risk index distributions are centered and have variance 1.

Each client's risk index X_i is then given as

$$X_i = \sqrt{\rho} \cdot Y + \sqrt{1-\rho} \cdot Z_i$$

with $i = 1,...,n$ if n is the number of clients in the portfolio where Y and Z_i are standard normally distributed.

We can now state

Theorem 2[211]*:*

In a homogenous portfolio containing n clients each having an exposure of $E = 1/n$ and risk index correlations $0 \le \rho < 1$ the α-percentile[212] of the asymptotic portfolio loss distribution is given as

$$\lim_{n\to\infty} L(\alpha; p, E(n), \rho, \lambda) = L(\alpha; p, \rho, \lambda) = \lambda \cdot \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}} \right).$$

Proof:

By definition of the model, client i defaults if

$$X_i = \sqrt{\rho} \cdot Y + \sqrt{1-\rho} \cdot Z_i \le \Phi^{-1}(p)$$
$$\Leftrightarrow Z_i \le \frac{\Phi^{-1}(p) - \sqrt{\rho} \cdot Y}{\sqrt{1-\rho}}$$

for $i = 1,...,n$.

Hence, client i's probability of default conditional to Y is given as

---

$$n = \left\lfloor \frac{\rho - 1}{\rho} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the integer part of x. Since we are interested in asymptotic results for $n \to \infty$, we will only consider non-negative correlations $\rho \ge 0$.

[210] Note that we do not assume recovery rates or loss given default rates as being fixed. They may be random with the same mean (not necessarily the same distribution) being independent from all other random variables in the model such as systematic and idiosyncratic risk factors.

[211] A similar version of Theorem 2 was probably first proved by Vasicek 1991. For other proofs or proof concepts see Gersbach / Wehrspohn 2001, Overbeck / Stahl 1999, Finger 1998.

[212] The α-percentile of the portfolio loss distribution is defined as

$$L(\alpha) = \inf\{l : P(\text{loss} \le l) \ge \alpha\}.$$

$L(\alpha)$ is also called the portfolio value at risk at confidence level α.

$$\mathbf{P}\{\text{client } i \text{ defaults} \mid Y\} = \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}\cdot Y}{\sqrt{1-\rho}}\right)$$

because Z_i is standard normally distributed for $i = 1,...,n$.

Moreover, since the idiosyncratic components Z_i of clients' risk indices are stochastically independent, it follows from the law of large numbers that the percentage of clients defaulting in the portfolio given Y is equal to their conditional probability of default almost surely[213] if $n \to \infty$.

Note that asymptotically the number of defaulting clients goes to infinity as well if the conditional probability is positive. Thus, again by the law of large numbers, the portfolio loss conditional to Y is equal to

$$\text{Loss} \mid Y = \lambda \cdot \Phi\left(\frac{\Phi^{-1}(p) - \sqrt{\rho}\cdot Y}{\sqrt{1-\rho}}\right)$$

since the individual loss given default rates are stochastically independent and limited with the same mean λ.

The inverse cumulative loss distribution function can immediately be derived from this expression for portfolio losses because the systematic risk factor Y is the only remaining random component in portfolio losses and the conditional portfolio loss given Y is monotonously decreasing in Y. The α-percentile of the portfolio loss distribution, therefore, maps one-to-one to the $(1 - \alpha)$-percentile of the distribution of Y. The $(1 - \alpha)$-percentile of Y is given by $\Phi^{-1}(1-\alpha)$ because Y is standard normally distributed. This completes the proof.

□

A few special cases are of interest:

- For $\rho = 0$, we have

$$L(\alpha; p, 0, \lambda) = \lambda \cdot \Phi\left(\Phi^{-1}(p)\right) = \lambda \cdot p,$$

  i.e. the portfolio loss distribution is constant at the expected loss as a direct consequence of the law of large numbers. This is due to the fact that in the normal correlation model uncorrelated risk indices are automatically independent. This is a very special feature of

[213] I.e. with probability 1.

the normal distribution, though, that does not carry over to the more general case that we will discuss in the next section.

- For $\rho > 0$, we have

$$\lim_{\alpha \to 1} L(\alpha; p, \rho, \lambda) = \lambda ,$$

i.e. for any positive dependencies in the model, the loss distribution converges in the confidence level α towards the loss of the entire portfolio.

- For $\rho \to 1$ the loss distribution converges towards the discontinuous step function l given by

$$l = \begin{cases} 0, & \text{if } \alpha \le 1 - p \\ \lambda, & \text{else} \end{cases} .$$

Figure 35 illustrates the loss distributions of homogenous portfolios for various correlations and a default probability of 0.5%. Again, it is obvious that the degree of dependence between clients, i.e. the size of correlations in this model, very strongly influences portfolio risk, especially for high confidence levels α.

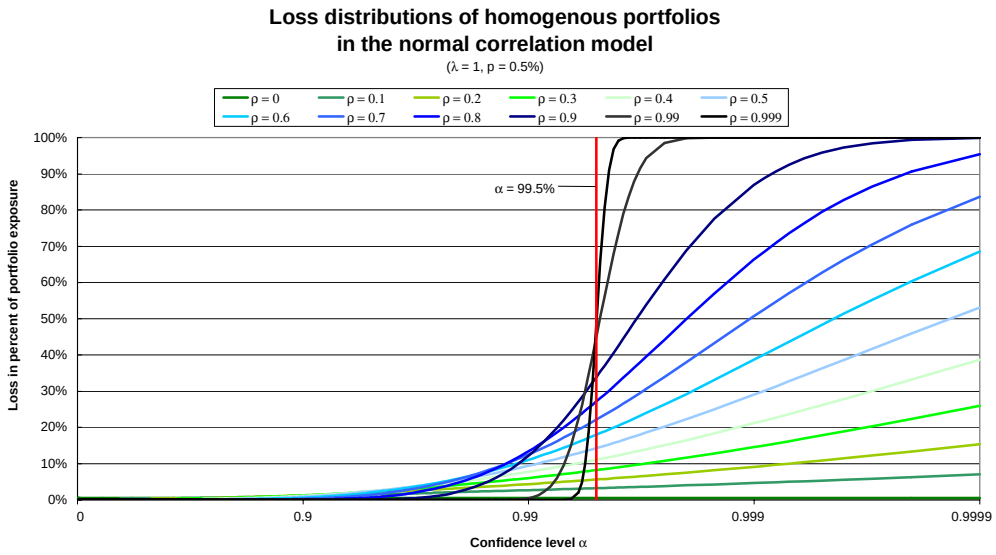


Figure 35: Loss distributions of homogenous portfolios in the normal correlation model 1

We express another interesting feature of homogenous portfolios as

Theorem 3:

Assume a homogenous portfolio as in Theorem 2. Let $\rho_1 < \rho_2$. Then there exists a confidence level α^* such that

$$L(\alpha; p, \rho_2, \lambda) < L(\alpha; p, \rho_1, \lambda) \quad \text{if } \alpha < \alpha^*$$
$$L(\alpha; p, \rho_2, \lambda) = L(\alpha; p, \rho_1, \lambda) \quad \text{if } \alpha = \alpha^*$$
$$L(\alpha; p, \rho_2, \lambda) > L(\alpha; p, \rho_1, \lambda) \quad \text{if } \alpha > \alpha^*$$

Theorem 3 states that increasing correlations and, thus, increasing dependencies and concentrations in the portfolio imply lower portfolio risk for low confidence levels α and higher portfolio risk for high confidence levels in terms of the respective percentiles of the loss distribution[214]. In other words, high correlations reduce portfolio risk most of the time, but also increase the probability of extreme losses.

Proof:

Since $\Phi(\cdot)$ is monotonously increasing, the statement $L(\alpha; p, \rho_2, \lambda) \begin{matrix} < \\ = \\ > \end{matrix} L(\alpha; p, \rho_1, \lambda)$

is equivalent to

$$F(\alpha; \rho_2) := \frac{\Phi^{-1}(p) - \sqrt{\rho_2} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_2}} \begin{matrix} < \\ = \\ > \end{matrix} \frac{\Phi^{-1}(p) - \sqrt{\rho_1} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho_1}} = F(\alpha; \rho_1)$$

with $F(\alpha; \rho_2) = F(\alpha; \rho_1)$ if and only if

$$\alpha^* := \alpha = 1 - \Phi\left(\frac{\left(\sqrt{1-\rho_2} - \sqrt{1-\rho_1}\right) \cdot \Phi^{-1}(p)}{\sqrt{1-\rho_2} \cdot \sqrt{\rho_1} - \sqrt{1-\rho_1} \cdot \sqrt{\rho_2}} \right) \in (0;1).$$

The theorem then follows from[215]

$$F'(\alpha^*; \rho_2) = \frac{\sqrt{\rho_2}}{\sqrt{1-\rho_2}} \cdot \frac{1}{\varphi\left(\Phi^{-1}(1-\alpha^*)\right)} > \frac{\sqrt{\rho_1}}{\sqrt{1-\rho_1}} \cdot \frac{1}{\varphi\left(\Phi^{-1}(1-\alpha^*)\right)} = F'(\alpha^*; \rho_1)$$

because $F(\alpha; \rho)$ is continuous in α for all $0 \le \rho < 1$.

In other words, $F(\alpha; \rho_1)$ and $F(\alpha; \rho_2)$ intersect only in $\alpha = 0$, $\alpha = 1$ and $\alpha = \alpha^*$ and for the slope we have $F'(\alpha^*; \rho_2) > F'(\alpha^*; \rho_1)$ in α^*, which implies that $F(\alpha; \rho_2) < F(\alpha; \rho_1)$ for

[214] See above footnote 212.

[215] Here $\varphi(\cdot)$ is the standard normal density function. It was used that for a continuous invertible function $f : X \to Y,\ x \mapsto y$ we have $\left(f^{-1}(y)\right)' = \frac{1}{f'\left(f^{-1}(y)\right)}$.

$\alpha < \alpha^*$ and $F(\alpha;\rho_2) > F(\alpha;\rho_1)$ for $\alpha > \alpha^*$. The theorem then follows from the monotonicity of $\Phi(\cdot)$.

□

Note that α^* is increasing in ρ_1 and ρ_2 and that

$$\alpha^*_{\max} = \lim_{\rho_1 \to 1} \alpha^*(\rho_1, \rho_2) = \lim_{\rho_2 \to 1} \alpha^*(\rho_1, \rho_2) = 1 - p\,.$$

Figure 36 gives a visual example of Theorem 3 and its implications.

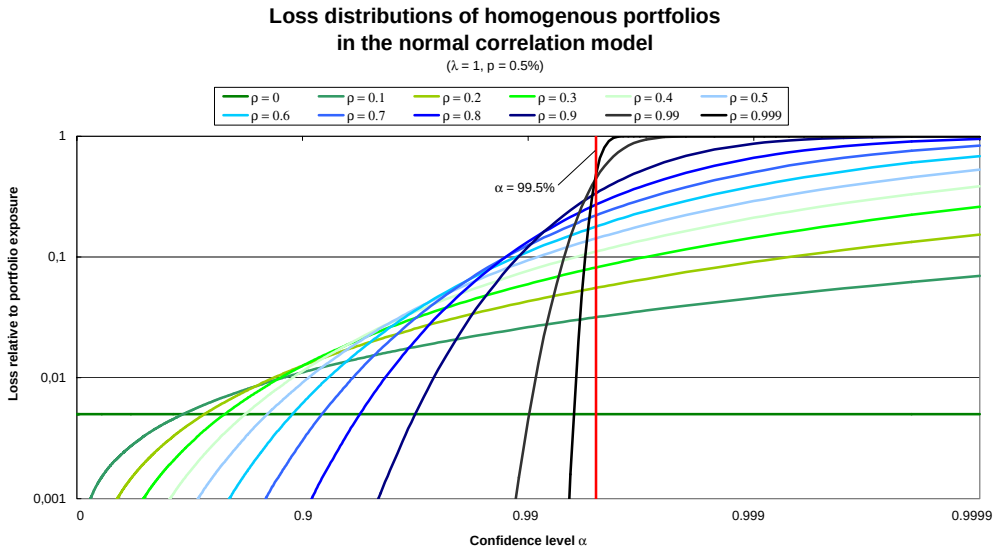


Figure 36: Loss distributions of homogenous portfolios in the normal correlation model 2

2. The generalized correlation model (CRE model)

While being economically intuitive, a major drawback of the normal correlation model is the somewhat arbitrary choice of the multivariate normal distribution to describe the joint movements of clients' individual risk indices. Historical reasons certainly were dominant in this selection because normal distributions appear as finite dimensional marginal distributions of the log-returns of the geometric Brownian motion, the standard model of continuous stochastic processes. This is used for example in the classic Black-Scholes-Merton model and is also by far the best understood continuous multivariate distribution.

In the present discussion, however, a typical criticism of the normal distribution is that it is not well adapted to the specific features of much financial data. This assessment refers especially to the phenomenon that many empirical distributions have long tails, i.e. that large de-

viations from the mean of a distribution are observed much more frequently than one would expect if the underlying distribution were normal.

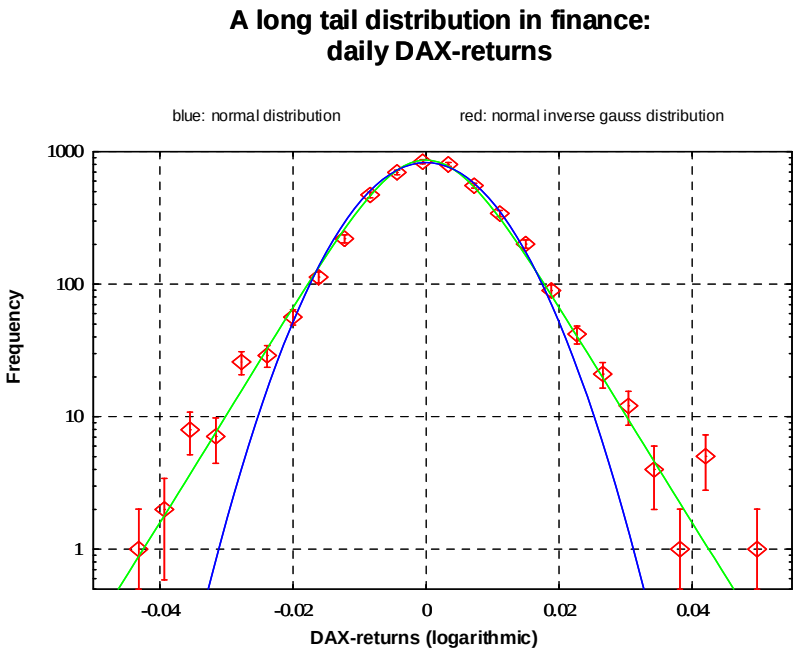


Figure 37: An empirical long tail distribution in finance: DAX-returns

Figure 37 gives an example of the normal and a long tail distribution fitted to the same financial data, daily DAX-returns from January 1, 1991 to November 30, 2000. It is apparent that the long tail distribution[216] is much better adapted to the frequency of extreme returns than the normal distribution while both distributions are quite similar in the region of small deviations from the mean.

In the normal correlation model, two things were fundamental: the marginal distributions that were needed to calculate clients' default and transition thresholds and the correlation matrix of clients' risks indices. In order to extend the model, note that a multivariate distribution is in general not uniquely determined by its marginal distributions and its correlation structure. However, exceptions in that respect are spherical and elliptical distributions.

A distribution D is called spherical[217] if it is invariant under orthogonal transformations, i.e. if for a random vector $X \in \mathbf{R}^n$ with $X \sim D$ and any orthogonal map $U \in \mathbf{R}^{n\times n}$ the equation

$$\mathcal{L}(X) = \mathcal{L}(UX)$$

holds[218]. If D has a density d than this definition is equivalent to saying that d is constant on spheres.[219]

[216] We chose the normal inverse gauss distribution in the example, a family of distributions where the length of the tails can be continuously adapted due to a further parameter besides mean and covariance matrix. See below for details.

[217] Or 'spherically symmetric'.

Let $\boldsymbol{S}$ be the family of all spherical distributions. A distribution D is called elliptical[220] if it is an affine linear transformation of a spherical distribution[221], i.e. if for a random vector $X \in \mathbf{R}^n$ with $X \sim D$ and a random vector $Y \in \mathbf{R}^n$ with $\mathcal{L}(Y) \in \boldsymbol{S}$ there exists $\mu \in \mathbf{R}^n$ and $A \in \mathbf{R}^{n \times n}$ such that

$$X = \mu + A \cdot Y .$$

The best known example of spherical or elliptical distributions, respectively, is the family of multivariate normal distributions so prominent in the normal correlation model. Figure 38 shows plots of densities and their contours of the bivariate normal distribution for correlations $\rho = 0$ (the spherical case) and for $\rho = 75\%$ (an elliptical case).

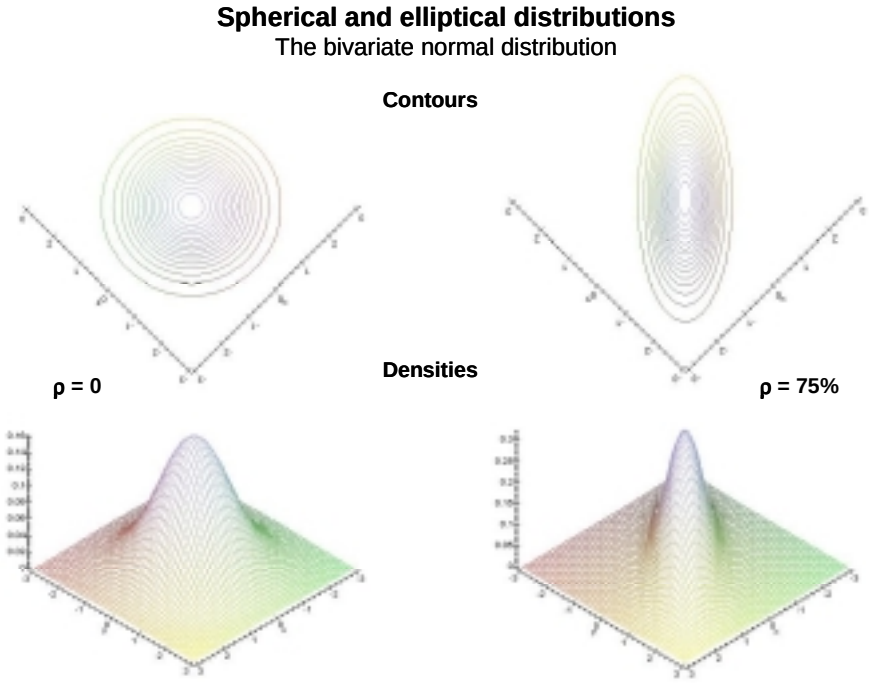


Figure 38: Spherical and elliptical distributions

[218] $\mathcal{L}(X)$ denotes the law of *X*. The expression denotes that the distributions of *X* and of *UX* are equal.

[219] Fang et al. 1989, definition 2.1., p. 29.

[220] Or 'elliptically symmetric'.

[221] Fang et al. 1989, definition 2.2., p. 31.

Elliptical distributions are an interesting generalization of the normal distribution in the correlation model because a multivariate elliptical distribution is uniquely determined by its univariate marginals, its mean and its covariance matrix since the type of all marginals is the same[222].

Not all symmetric univariate distributions are possible as marginal distributions of an elliptical distribution in $\mathbf{R}^n$ for any $n \in \mathbf{N}$. It can be shown, however, that a univariate distribution D is the marginal distribution of a spherical distribution in $\mathbf{R}^n$ for any $n \in \mathbf{N}$ if and only if it is a variance mixture of centered normals[223]. Hence[224], D can be defined by its density function

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{s}} \exp\left(-\frac{x^2}{2s}\right) dW(s)$$

where the weight or mixing distribution W only takes values on $(0,\infty)$, i.e. a variance mixture of normals is a normal distribution with random variance. This definition immediately implies that a random variable $X \sim D$ can be written as

$$X = \sqrt{w} \cdot Y$$

where Y is standard normally distributed, $w \sim W$, and Y and w are stochastically independent.

Example 1

A well known example of a normal variance mixture is the Student-t distribution with n degrees of freedom. Here the mixing distribution w is given as

$$w = \frac{n}{\vartheta}$$

where $\vartheta \sim \chi_n^2$.

Example 2

A more flexible family of mixture distributions is the generalized hyperbolic distribution. The one-dimensional centered and symmetric version of the generalized hyperbolic distribution has three free parameters λ, α, δ and is defined by its Lebesgue-density

$$gh(x;\lambda,\alpha,\delta) = a(\lambda,\alpha,\delta) \cdot \left(\delta - x^2\right)^{(\lambda-1/2)/2} \mathbf{K}_{\lambda-1/2}\left(\alpha\sqrt{\delta^2 + x^2}\right)$$

222 Cf. Embrechts et al. 1999, p. 11.

223 Fang et. al, 1989, theorem 2.21, p. 48. Note that there exist univariate distributions that are no variance mixtures of normals that can be marginals of spherical distributions for some, but not all $n \in \mathbf{N}$.

224 Note that any mixture of normals has a density with respect to Lebesgue measure.

with

$$a(\lambda,\alpha,\delta)=\frac{\alpha^{1/2}}{\sqrt{2\pi}\cdot\delta^{\lambda}\cdot\mathbf{K}_{\lambda}(\delta\lambda)}$$

where $\mathbf{K}_{\lambda}(\cdot)$ is a modified Bessel function of the third kind with index λ and $x\in\mathbf{R}$. Alternatively, the generalized hyperbolic distribution can be defined by its mixing distribution. This is the generalized inverse Gauss distribution with density

$$gig(x;\psi,\chi)=\frac{(\psi/\chi)^{\lambda/2}}{\mathbf{K}_{\lambda}\left(\sqrt{\psi\chi}\right)}x^{\lambda-1}\cdot\exp\left(-\frac{1}{2}\left(\frac{\chi}{x}+\psi x\right)\right)$$

for $x>0$ and $\chi=\delta^2$ and $\psi=\alpha^2$.

The generalized hyperbolic distribution is continuous in its parameters and has the normal and the t-distribution as limiting cases:

For $\alpha,\delta\to\infty$, $\frac{\delta}{\alpha}\to\sigma^2$ and any given λ, the generalized hyperbolic distribution converges towards $N\left(0,\sigma^2\right)$.[225]

On the other hand, for $\alpha=0$, $\delta=\sqrt{\nu}$ and $\lambda=-\nu/2$ it is equal to the t-distribution with ν degrees of freedom.[226]

Example 3

An interesting special case of the generalized hyperbolic distribution is the normal inverse Gauss distribution (NIG). It is obtained for $\lambda=-1/2$ and has the inverse Gauss distribution

$$ig(x;\psi,\chi)=\sqrt{\frac{\chi}{2\cdot\pi\cdot x^3}}\cdot\exp\left(-\frac{1}{2}\left(\frac{\chi}{x}+\psi x\right)+\sqrt{\psi\chi}\right)$$

as mixing distribution.

The NIG is particularly interesting as an alternative for the normal distribution in the correlation model because it is not only infinitely divisible[227], but also closed under convolution. Hence, similar to the normal distribution, it generates a Lévy-motion whose finite dimensional marginals are all NIG-distributed[228]. Therefore, the intuitive interpretation in the Va-

225 Cf. Prause 1999, p. 3.

226 Cf. Prause 1999, p. 5.

227 As is any generalized hyperbolic distribution. Cf. Barndorff-Nielsen and Halgreen 1977.

228 Cf. Eberlein et al. 1998, p. 6f., who use the Lévy-motion generated by the NIG instead of the classical geometric Brownian motion to model financial price processes.

sicek-Kealhofer model and in Credit Metrics of the normal distribution as the marginal distribution of an asset return process could be maintained in a one-to-one fashion if the normal distribution is replaced by the NIG.

Example 4

A very simple family of mixtures of normals are finite mixture distributions. They are obtained if the mixing distribution W takes on only finitely many values with positive probability, i.e. if there exist real numbers $w_1,...,w_n > 0$ such that

$$\mathbf{P}\left(w \in \{w_1,...,w_n\}\right) = 1$$

if $w \sim W$.

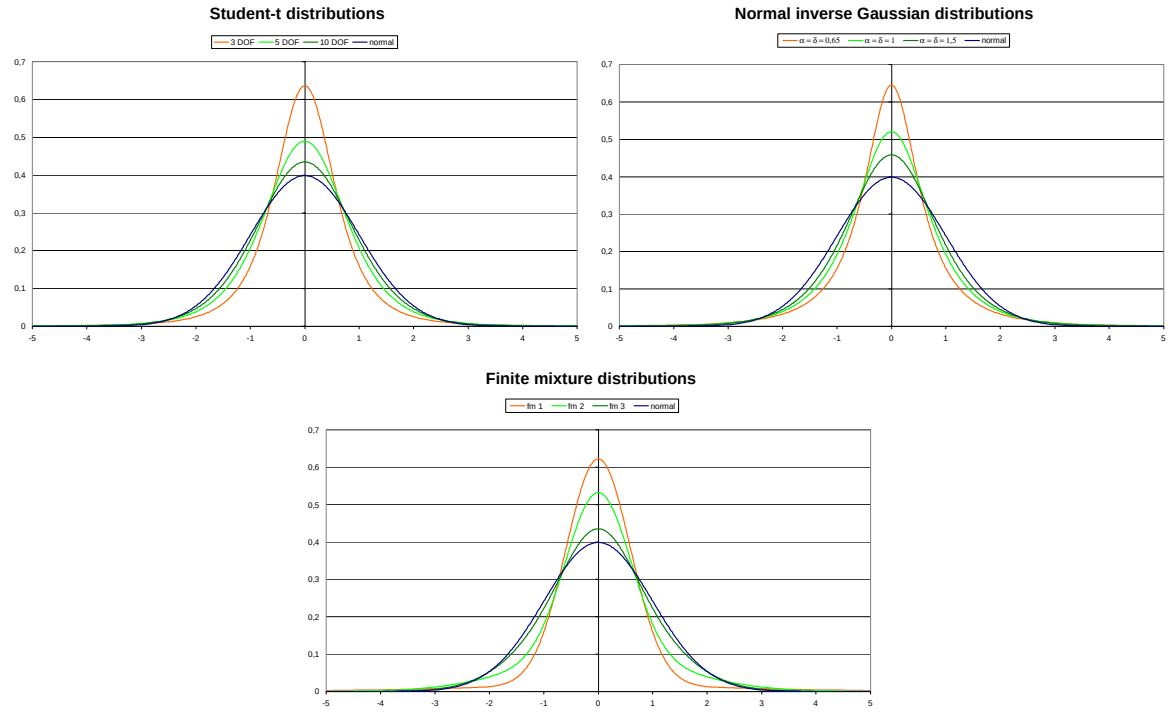


Figure 39: Densities of normal variance mixture distributions[229]

Figure 39 gives an impression of the shape of the densities of some normal variance mixtures. All distributions are standardized to variance 1. Besides the flexibility of deformation, the distributions differ above all in tail behavior.

[229] The mixing distributions of the finite mixtures in the graph are defined as follows:

fm 1: $\mathbf{P}\{w = 0.35\} = 0.9, \quad \mathbf{P}\{w = 6.85\} = 0.1$

fm 2: $\mathbf{P}\{w = 0.35\} = 0.65, \quad \mathbf{P}\{w = 2.21\} = 0.35$

fm 3: $\mathbf{P}\{w = 0.35\} = 0.225, \quad \mathbf{P}\{w = 1.19\} = 0.775$

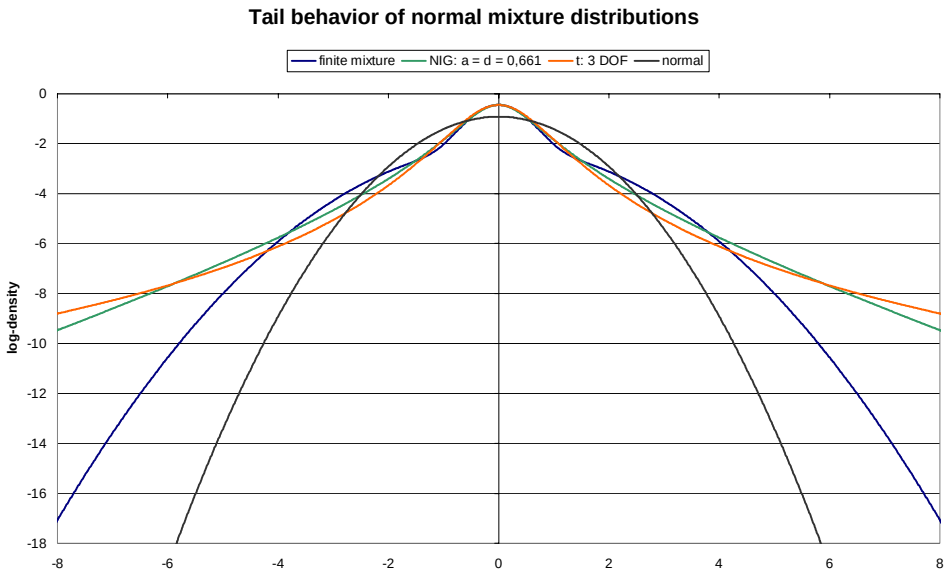


Figure 40: Tail behavior of normal mixture distributions

Figure 40 shows that NIG and t-distributions have exponentially decreasing tails while tails of finite mixture distributions and the normal distribution decrease of order[230] $O\left(e^{-x^2}\right)$.

However, interpreting the illustration above, we can prove the following

Theorem 4 and definition:

Let *f* be the density function of a distribution *F* with expectation μ and variance σ^2. We say that *F* has long tails compared to the normal distribution if and only if

$$\lim_{x\to\infty}\frac{f(x)}{\varphi_{\mu,\sigma^2}(x)}>1$$

where φ_{μ,σ^2} is the density of the normal distribution with expectation μ and variance σ^2.

Let *D* be a normal variance mixture with non-degenerate mixing distribution. Then *D* has long tails compared to the normal distribution.

Proof:

Without loss of generality, we can assume that $\mu=0$ and $\sigma^2=1$. Let *W* be the mixing distribution of *D* and $w\sim W$. Then there exists an $s>1$ such that $p:=\mathbf{P}\{w\geq s\}>0$.

Let f be the density of *D*. Then we have

[230] A function *f* is of order $O\left(g\left(x\right)\right)$ if and only if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=c\neq 0$, i.e. $O(\cdot)$ describes the asymptotic expansion of *f*.

$$\lim_{x\to\infty}\frac{f(x)}{\varphi(x)} > \lim_{x\to\infty}\frac{p\cdot\frac{1}{\sqrt{s}}\cdot\varphi\left(\frac{x}{\sqrt{s}}\right)}{\varphi(x)} = \lim_{x\to\infty}\frac{p\cdot e^{-\frac{x^2}{2s}}}{\sqrt{s}\cdot e^{-\frac{x^2}{2}}} = \lim_{x\to\infty}\frac{p}{\sqrt{s}}e^{\frac{x^2}{2}\left(1-\frac{1}{s}\right)} = \infty .$$

□

We say that a model defines a representation of the generalized correlation model, if its risk index distribution is a variance mixture of normals. According to the choice of the risk index distribution in the generalized correlation model, we will also speak of the Student-t-correlation model, the finite mixture correlation model etc. The normal correlation model is just a special case in this framework. It is obtained for $w \equiv c$ for some constant $c > 0$.

To better understand the impact of the choice of clients' risk index distributions on the resulting portfolio risk, we look at

a) Homogenous Portfolios

because we can derive analytic loss distributions for this special class of portfolios making it particularly convenient to compare the results.

As in the normal correlation model, we define a homogenous portfolio as consisting only of identical clients in terms of probabilities of default p, exposure E, risk index correlations[231] ρ, and expected loss given default λ[232]. Without loss of generality, we may assume that all risk index distributions are centered and have variance 1.

Let D be the distribution of risk indices. We assume that D is a mixture of normals with mixing distribution W, such that

$$\sqrt{w}\cdot X \sim D$$

if

$$w \sim W \text{ and } X \sim \mathrm{N}(0,1).$$

In the generalized correlation model, in a homogenous portfolio each client's risk index X_i is then given as

$$X_i = \sqrt{w}\cdot\left(\sqrt{\rho}\cdot Y + \sqrt{1-\rho}\cdot Z_i\right)$$

[231] See above footnote 209.

[232] Again we do not assume recovery rates or loss given default rates as being fixed. They may be random with the same mean (not necessarily the same distribution) being independent from all other random variables in the model such as systematic and idiosyncratic risk factors.

with $i = 1,...,n$ if n is the number of clients in the portfolio where Y and Z_i are standard normally distributed.

To facilitate the exposition of the results,

- let F be the cumulative distribution function of D,
- let Φ be the cumulative distribution function of the standard normal distribution,
- let L be the loss distribution of the portfolio under consideration, i.e. the cumulative distribution function of portfolio losses.

We begin with the derivation of the portfolio loss distribution of homogenous portfolios in the generalized correlation model, then calculate the density of the portfolio loss distribution and compare the results for the normal and generalized correlation model. Finally, we show how the approach can be extended to more complex forms of homogenous portfolios.

(1) Portfolio loss distribution

Theorem 5:
In the generalized correlation model, in a homogenous portfolio containing n clients each having an exposure of $E = 1/n$ and risk index correlations $0 < \rho < 1$ the asymptotic portfolio loss distribution is given as

$$\begin{aligned} L(l; p, \rho, \lambda) &= \lim_{n\to\infty} L(l; p, E(n), \rho, \lambda) = \mathbf{P}\{\text{Loss} \le l\} \\ &= 1 - \mathbf{E}_w\left(\Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}} \right) \right) \end{aligned}$$

where $\mathbf{E}_w(\cdot)$ is the expectation functional with respect to w.

Proof:
The first part of the proof is similar to the proof of Theorem 2.

By definition of the model, client i defaults if

$$\begin{aligned} X_i &= \sqrt{w}\cdot\left(\sqrt{\rho}\cdot Y + \sqrt{1-\rho}\cdot Z_i\right) \le F^{-1}(p) \\ \Leftrightarrow Z_i &\le \frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}} \end{aligned}$$

for $i = 1,...,n$.

Hence, client i's probability of default conditional to w and Y is given as

$$\mathbf{P}\{\text{client } i \text{ defaults} \mid w,Y\} = \Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)$$

because Z_i is standard normally distributed for $i = 1,...,n$.

Moreover, since the idiosyncratic components Z_i of clients' risk indices are stochastically independent, it follows from the law of large numbers that the percentage of clients defaulting in the portfolio given *w* and *Y* is equal to their conditional probability of default with probability one if $n \to \infty$.

Note that asymptotically the number of defaulting clients goes to infinity as well if the conditional probability is positive. Thus, again by the law of large numbers, the portfolio loss conditional to *w* and *Y* is equal to

$$\text{Loss} \mid w,Y = \lambda\cdot\Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)$$

since the individual loss given default rates are stochastically independent and limited with the same mean λ.

The unconditional portfolio loss distribution is, therefore, given as

$$\begin{aligned}
\mathbf{P}\{\text{Loss} \le l\} &= \mathbf{P}\left\{\lambda\cdot\Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right) \le l\right\} \\
&= \mathbf{P}\left\{\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}} \le \Phi^{-1}(l/\lambda)\right\} \\
&= \mathbf{P}\left\{Y \ge \frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right\} \\
&= 1 - \mathbf{P}\left\{Y \le \frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right\} \\
&= 1 - \mathbf{E}_w\left(\mathbf{P}\left\{Y \le \frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}} \mid w\right\}\right) \\
&= 1 - \mathbf{E}_w\left(\Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right)\right)
\end{aligned}$$

□

The portfolio loss distribution is particularly easy to calculate for finite mixture distributions because in this case the expectation functional $\mathbf{E}_w(\cdot)$ is reduced to a simple sum. Let the mixing distribution W be a probability distribution on $\{w_1,...,w_k\}$ with $\mathbf{P}\{w=w_i\}=p_i$ for $i=1,...,k$. Then the portfolio loss distribution can be written as

$$L(l;p,\rho,\lambda)=1-\sum_{i=1}^{k} p_i \cdot \Phi\left(\frac{F^{-1}(p)-\sqrt{w_i}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w_i}\cdot\sqrt{\rho}}\right).$$

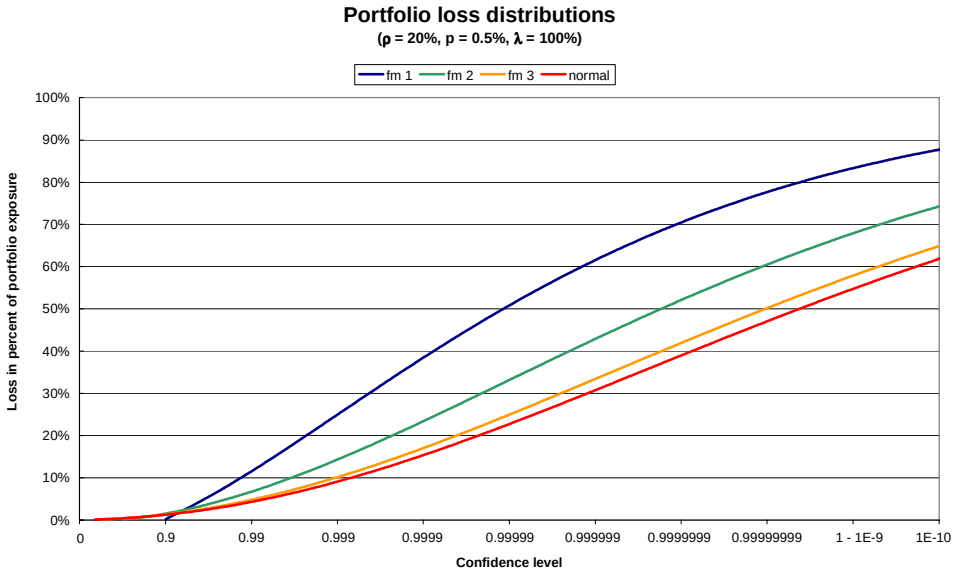


Figure 41: Portfolio loss distributions in the generalized correlation model based on finite mixture distributions

Figure 41 shows the loss distributions for a homogenous portfolio resulting from the finite mixture distributions in Figure 39. Note that the loss distributions in the generalized correlation model dominate the loss distribution in the normal correlation model at high confidence levels, in the example at confidence levels above 92.5% (fm 1), 84.8% (fm 2), and 79.9% (fm 3). We will prove this as a general result in Theorem 9 below.

For more complex mixing distributions the portfolio loss distribution can be calculated using numerical techniques or Monte Carlo integration.

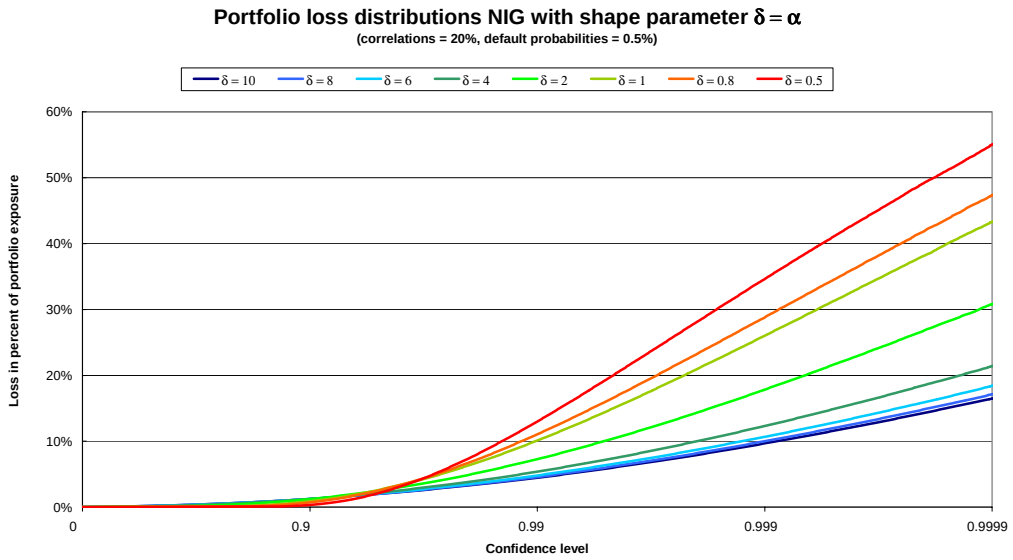


Figure 42: Portfolio loss distributions in the generalized correlation model based on normal inverse Gaussian distributions

It is a very important feature of the generalized correlation model that clients with uncorrelated risk indices are still dependent in their default behavior. We state this fact as

Theorem 6:

In the generalized correlation model, in a homogenous portfolio containing n clients each having an exposure of $E = 1/n$ and risk index correlations $\rho = 0$, the asymptotic portfolio loss distribution is given as

$$L(l;p,0,\lambda) = \begin{cases} \mathbf{P}\left\{\sqrt{w} \le \dfrac{F^{-1}(p)}{\Phi^{-1}(l/\lambda)}\right\} & \text{if } l < \lambda/2 \\ 1 - \mathbf{P}\left\{\sqrt{w} \le \dfrac{F^{-1}(p)}{\Phi^{-1}(l/\lambda)}\right\} & \text{if } l > \lambda/2 \\ 0 & \text{if } p > 1/2 \text{ and } l = \lambda/2 \\ 1 & \text{if } p \le 1/2 \text{ and } l = \lambda/2 \end{cases}$$

Proof:

If risk indices are uncorrelated, client i defaults if

$$X_i = \sqrt{w} \cdot Z_i \le F^{-1}(p)$$
$$\Leftrightarrow Z_i \le \frac{F^{-1}(p)}{\sqrt{w}} \quad .$$

Along the same lines as in Theorem 5 one can show that the portfolio loss distribution is then given as

$$\begin{aligned}\mathbf{P}\{\text{Loss} \le l\} &= \mathbf{P}\left\{\lambda \cdot \Phi\left(\frac{F^{-1}(p)}{\sqrt{w}}\right) \le l\right\} \\ &= \mathbf{P}\left\{\frac{F^{-1}(p)}{\sqrt{w}} \le \Phi^{-1}(l/\lambda)\right\} \\ &= \mathbf{P}\left\{F^{-1}(p) \le \sqrt{w} \cdot \Phi^{-1}(l/\lambda)\right\}\end{aligned}$$

which is equivalent to the formulation in the theorem since $\Phi^{-1}(l/\lambda) < 0$ if $l < \lambda/2$, $\Phi^{-1}(l/\lambda) > 0$ if $l > \lambda/2$, and $\Phi^{-1}(l/\lambda) = 0$ if $l = \lambda/2$.

□

Theorem 6 states that – other than in the normal correlation model[233] where $\rho = 0$ implies $\mathbf{P}\{\text{Loss} = p \cdot \lambda\} = 1$ – uncorrelated risk indices imply in the generalized correlation model a constant portfolio loss distribution only for $p = ½$.

For default probabilities $p < ½$ the loss distribution is not constant, but only takes values between 0 and $\lambda/2$. On the other hand, for default probabilities $p > ½$ the loss distribution is not constant either, and only takes values between $\lambda/2$ and λ.

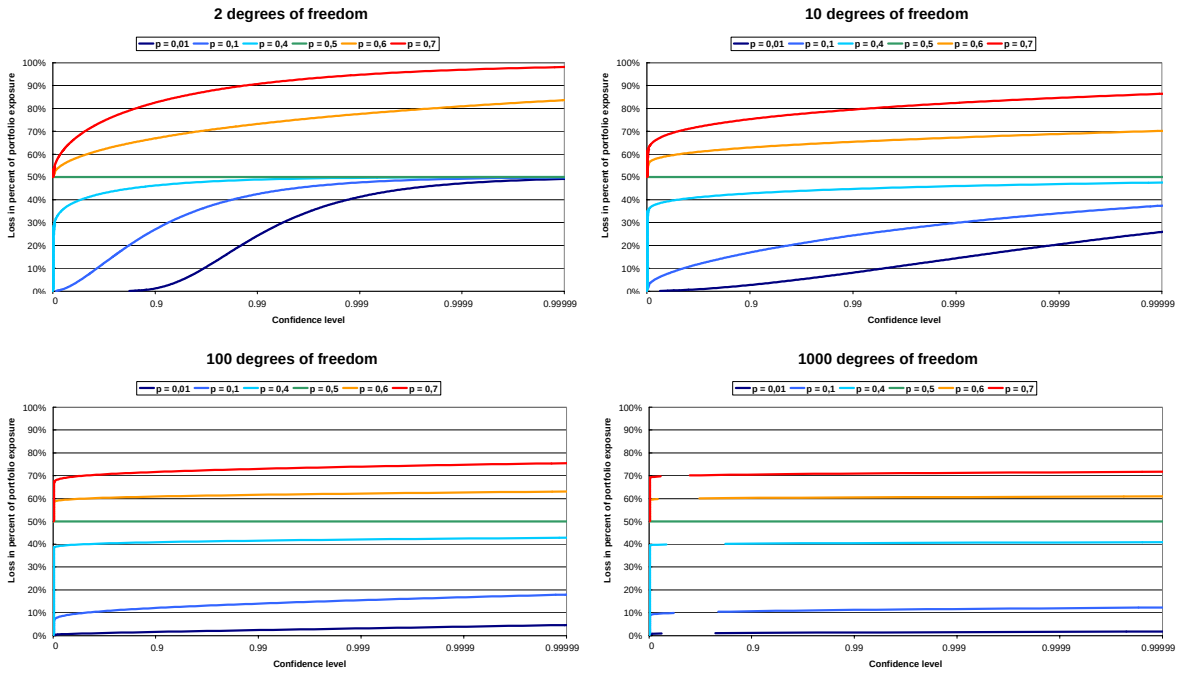


Figure 43: Loss distributions resulting from uncorrelated Student-t-distributed risk indices[234]

Figure 43 shows that loss distributions remain above or below $\lambda/2$ for the respective probabilities of default. If the number of the degrees of freedom of the Student-t-distribution tends

[233] The result for the normal correlation model follows from the theorem for $w \equiv 1$. In this case $F = \Phi$.

[234] The loss given default rate λ is set to 100%.

to infinity, i.e. if the t-distributed risk indices converge towards normally distributed risk indices, the portfolio loss distributions become more and more flat and converge towards $L(l; p,0,\lambda) \equiv p \cdot \lambda$ as we would expect from Theorem 6.

If the mixing distribution of the risk index distributions is discrete, as is the case at finite mixtures of normals, then the resulting portfolio loss distribution is also discrete if risk index correlations are zero (Figure 44). However, if the mixing distribution converges to a constant, the loss distribution again converges towards $L(l; p,0,\lambda) \equiv p \cdot \lambda$.

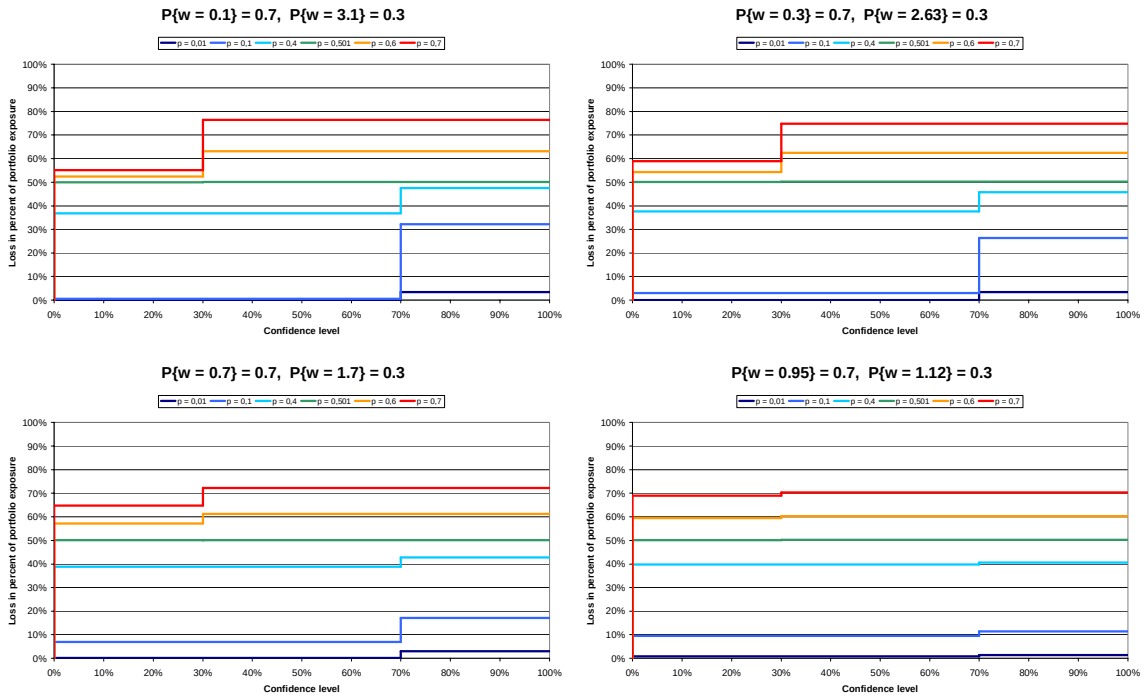


Figure 44: Loss distributions resulting from uncorrelated finite mixture distributed risk indices[235]

In the remaining case of perfect risk index correlations $\rho = 1$ the differences between the models disappear. Here clients' risk indices are given as

$$X_i = X = \sqrt{w} \cdot Y$$

so that all clients default simultaneously if

$$X = \sqrt{w} \cdot Y \le F^{-1}(p).$$

However, since $\sqrt{w} \cdot Y \sim F$ is

[235] The mixing distribution is a two-point distribution that is standardized to have expectation 1. The loss given default rate λ is set to 100%.

$$\mathbf{P}\left\{\sqrt{w}\cdot Y \le F^{-1}(p)\right\} = \mathbf{P}\left\{Y \le \Phi^{-1}(p)\right\} = p$$

independent of F.

(2) Portfolio loss density

In the previous section, we considered the asymptotic portfolio loss distribution of homogenous portfolios for an infinite number of clients in the portfolio. Due to the increasing number of clients, each single client's exposure converges to zero relative to the total portfolio exposure. Therefore, homogenous portfolios asymptotically do not contain exposure concentrations on specific clients or exposure point masses. This is a necessary condition for the portfolio loss distribution to have a Lebesgue density.

In the next theorem, we derive the portfolio loss density for risk index correlations $0 < \rho < 1$.

Theorem 7:

In the generalized correlation model, in a homogenous portfolio containing n clients each having an exposure of $E = 1/n$ and risk index correlations $0 < \rho < 1$, the density L_d of the asymptotic portfolio loss distribution is given as

$$L_d(l;\, p,\rho,\lambda) = \frac{\sqrt{1-\rho}}{\lambda\cdot\sqrt{\rho}}\cdot\frac{1}{\varphi\left(\Phi^{-1}(l/\lambda)\right)}\cdot\mathbf{E}_w\left(\varphi\left(\frac{F^{-1}(p)-\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right)\right)$$

where φ is the standard normal density.

Proof:

The portfolio loss density is defined as the first derivative of the cumulative distribution function of portfolio losses. Thus, we have

$$\begin{aligned} L_d(l;p,\rho,\lambda) &= \frac{d}{dl}L(l;p,\rho,\lambda) \\ &= \frac{d}{dl}\left(1-\mathbf{E}_w\left(\Phi\left(\frac{F^{-1}(p)-\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right)\right)\right) \\ &= -\mathbf{E}_w\left(\varphi\left(\frac{F^{-1}(p)-\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right)\cdot(-1)\cdot\frac{\sqrt{w}\cdot\sqrt{1-\rho}}{\sqrt{w}\cdot\sqrt{\rho}}\cdot\frac{1}{\varphi\left(\Phi^{-1}(l/\lambda)\right)}\cdot\frac{1}{\lambda}\right) \end{aligned}$$

since the standard normal density is continuous and integrable so that we may derive within the expectation functional $\mathbf{E}_w$.

□

Figure 45 shows loss densities for a model where the risk index distribution is a bi-mixture defined by its mixing distribution $\mathbf{P}\{w=0.2\}=0.5$ and $\mathbf{P}\{w=1.8\}=0.5$ for various default probabilities and risk index correlations.

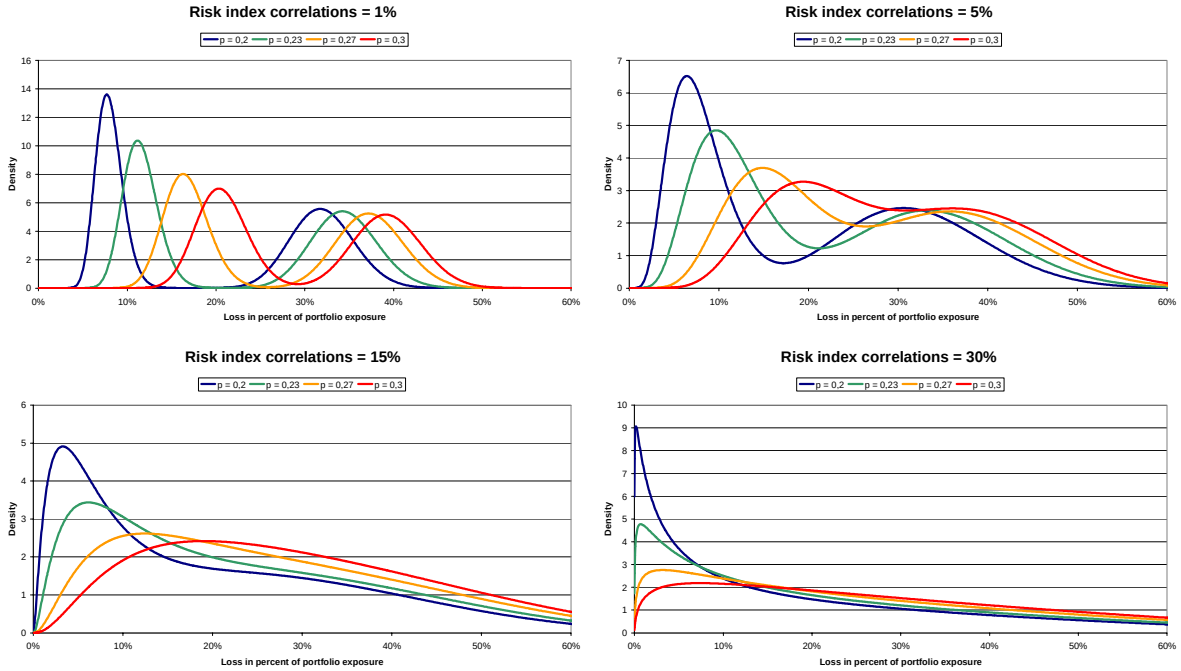


Figure 45: Portfolio loss densities in the bimixture correlation model

Note that the density function is not necessarily unimodal. The number of modes varies with default probabilities, risk index correlations and also with the type of mixing distribution.

We formulate this observation as

Theorem 8:

In the generalized correlation model, the number of modes of the asymptotic portfolio loss distribution of a homogenous portfolio with risk index correlations $0<\rho\le 1$, is smaller or equal to the cardinality of the support of the mixing distribution of the risk index distribution.

Proof:

It can be shown by derivation of the portfolio loss density of the normal correlation model[236], that the portfolio loss distribution is unimodal in this model (see also Figure 46) if $0<\rho\le 1/2$. It is bimodal if $1/2<\rho<1$ with peaks at 0 and the maximum loss.

The theorem then follows immediately from the fact that the portfolio loss distribution in the generalized correlation model is the convex combination of loss distributions in the normal correlation model. □

[236] The portfolio loss density of the normal correlation model is obtained by setting $w \equiv c$ for some constant c in Theorem 8.

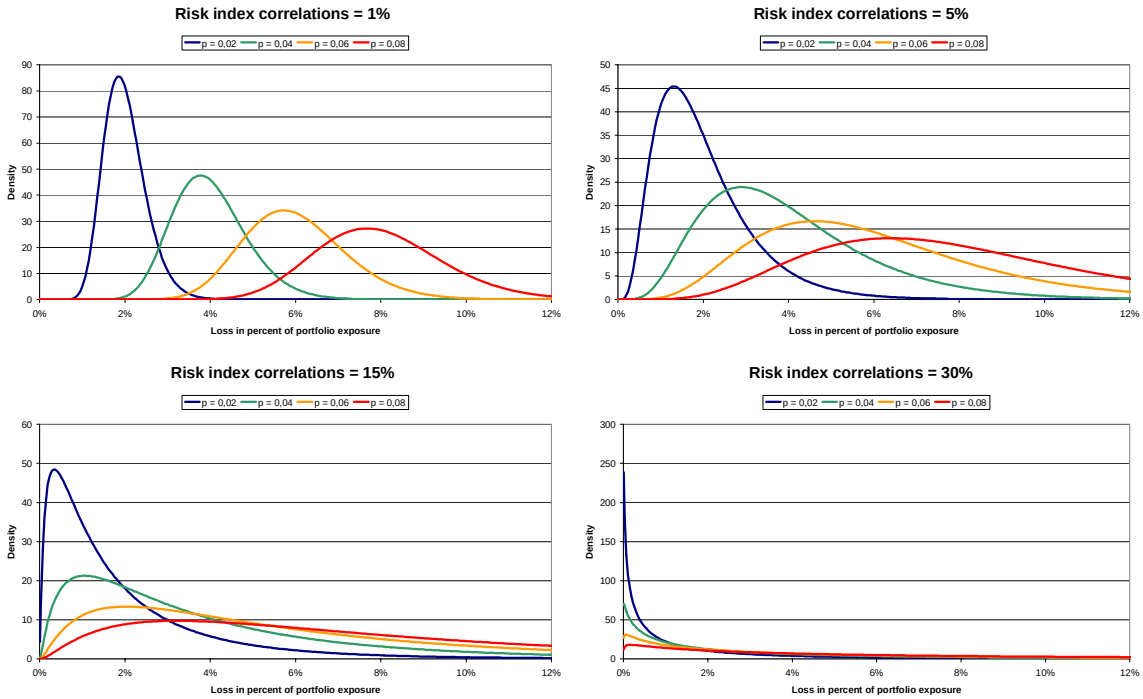


Figure 46: Portfolio loss densities in the normal correlation model

Figure 47 gives an example of trimodal loss densities in the trimixture correlation model.

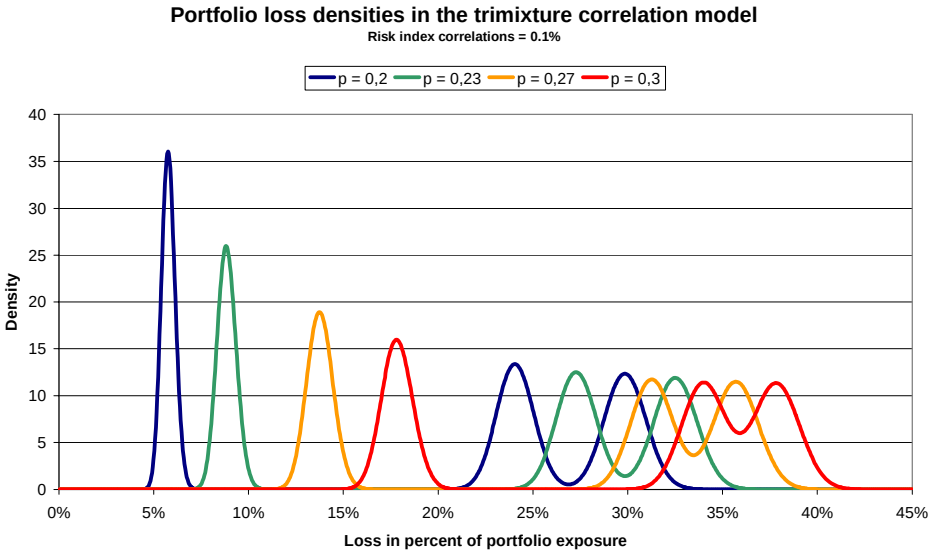


Figure 47: Trimodal portfolio loss densities in the trimixture correlation model[237]

We saw in the previous section, that the portfolio loss distribution is discrete for zero correlations in the finite mixture correlation model while it is Lebesgue absolutely continuous for

[237] The distributions differ only in the clients' default probability p and share the mixing distribution $\mathbf{P}\{w = 0.2\} = 1/3$, $\mathbf{P}\{w = 1\} = 1/3$, $\mathbf{P}\{w = 1.78\} = 1/3$. The modes of the original loss densities in the convex combinations are well visible and well separated.

positive risk index correlations. We would, therefore, expect that loss densities degenerate in this model if risk index correlations go to zero, a phenomenon that is illustrated by Figure 48.

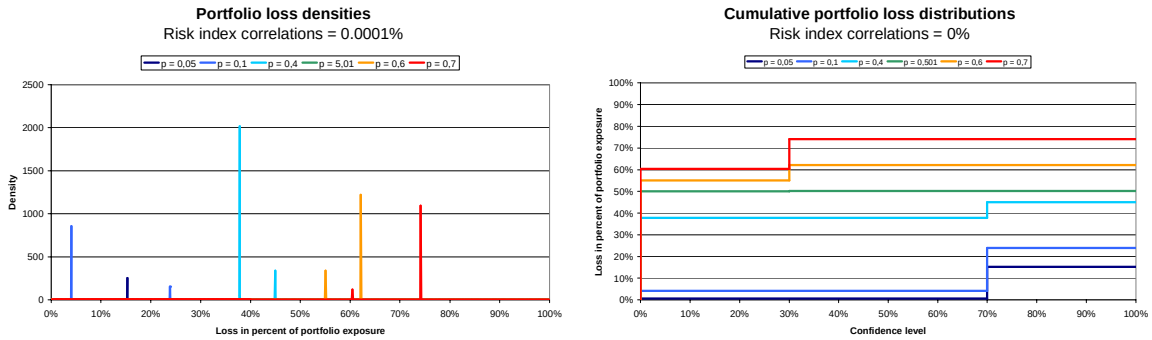


Figure 48: The low correlation effect in the finite mixture model[238]

For extremely low correlations, the loss density develops peaks at the discontinuity points of the loss distribution. The relative size of the peaks is approximately equal to the size of the point masses at the discontinuity points.

(3) Comparison of the normal and the generalized correlation model

So far the replacement of the normal distribution by a general distribution, determined by its marginal distributions and a correlation structure, was rather intuitively motivated as a general topic in the correlation model and by the fact that all of these distributions have long tails, a feature frequently found in finance (see section II.A.2 above). The next theorem gives a fundamental reason as to why the choice of the normal distribution as the distribution of risk indices might cause structural problems in the correlation model and why it should be carefully overthought. The theorem also shows that the analysis of homogenous portfolios can be extremely helpful in the understanding of the economic and model theoretic consequences of allegedly natural mathematical assumptions.

Theorem 9:

Let $L_N^{-1}(\cdot\,; p, \rho, \lambda)$ and $L_{G\backslash N}^{-1}(\cdot\,; p, \rho, \lambda)$ be the inverse cumulative distribution functions of the asymptotic portfolio loss distributions of a homogenous portfolio with risk index correlations $0 < \rho < 1$ in the generalized correlation model with a normal and a non-normal risk index distribution, respectively. Then there exists a confidence level α^* such that

$$L_{G\backslash N}^{-1}(\alpha; p, \rho, \lambda) > L_N^{-1}(\alpha; p, \rho, \lambda)$$

[238] The actual model displayed in the charts is a bimixture model with mixing distribution $\mathbf{P}\{w = 0.4\} = 0.7$ and $\mathbf{P}\{w = 2.4\} = 0.3$.

for all confidence levels $\alpha > \alpha^*$ and $p \neq \frac{1}{2}$.

Alternatively: Let $L_N(\cdot\,; p, \rho, \lambda)$ and $L_{G\setminus N}(\cdot\,; p, \rho, \lambda)$ be the cumulative distribution functions of the asymptotic portfolio loss distributions of the same homogenous portfolio with risk index correlations $0 < \rho < 1$ in the generalized correlation model with a normal and a non-normal risk index distribution, respectively. Then there exists a portfolio loss l^* such that

$$L_{G\setminus N}(l; p, \rho, \lambda) < L_N(l; p, \rho, \lambda)$$

for all portfolio losses $l > l^*$.

Theorem 9 states that for high confidence levels the normal correlation model reports less risk in any homogenous portfolio than any other correlation model (see Figure 49). I.e. at the high confidence levels risk managers are interested in, the normal correlation model is not robust against misspecifications in the risk index distribution.

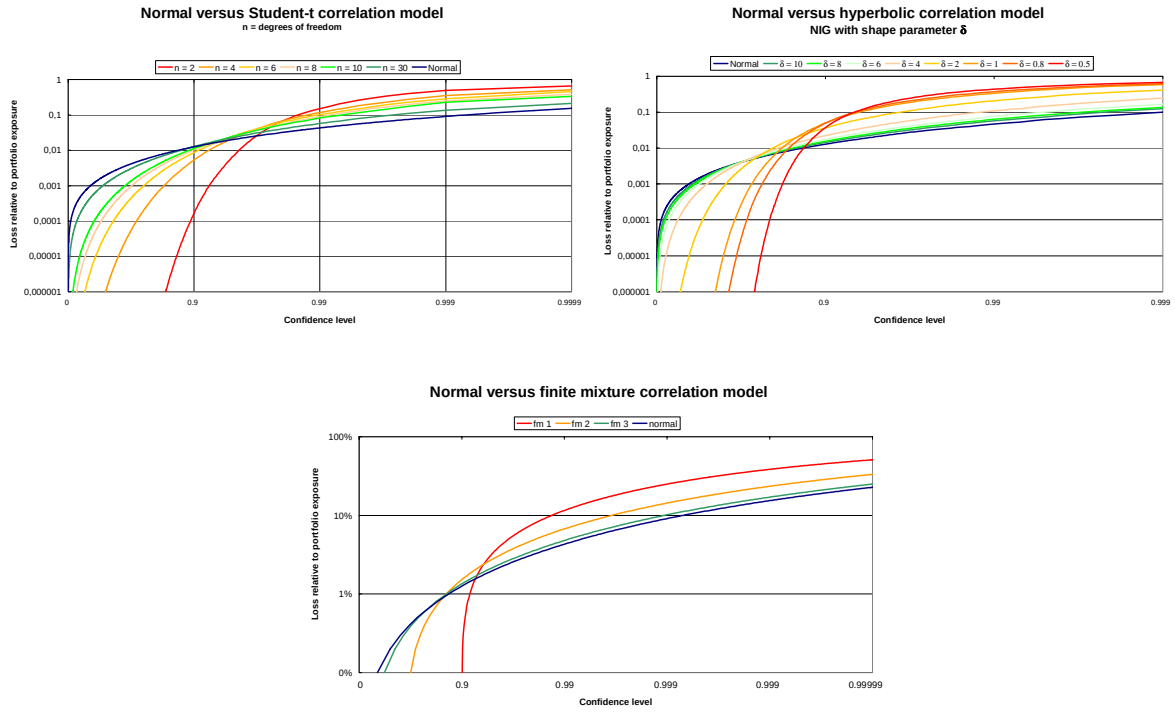


Figure 49: Portfolio loss distributions in the normal versus the generalized correlation model

Proof:

Let $L_{d,N}(\cdot\,; p, \rho, \lambda)$ and $L_{d,G\setminus N}(\cdot\,; p, \rho, \lambda)$ be the portfolio loss densities in the correlation model with a normal and a non-normal risk index distribution. We show that a portfolio loss l^* exists so that $L_{d,N}(l\,; p, \rho, \lambda) < L_{d,G\setminus N}(l; p, \rho, \lambda)$ for all portfolio losses $l > l^*$ (see Figure

50). This implies that $L_{G\backslash N}(l;p,\rho,\lambda) < L_N(l;p,\rho,\lambda)$ and, thus, the second version of the theorem because $L_{G\backslash N}(1;p,\rho,\lambda) = L_N(1;p,\rho,\lambda) = \lambda$.

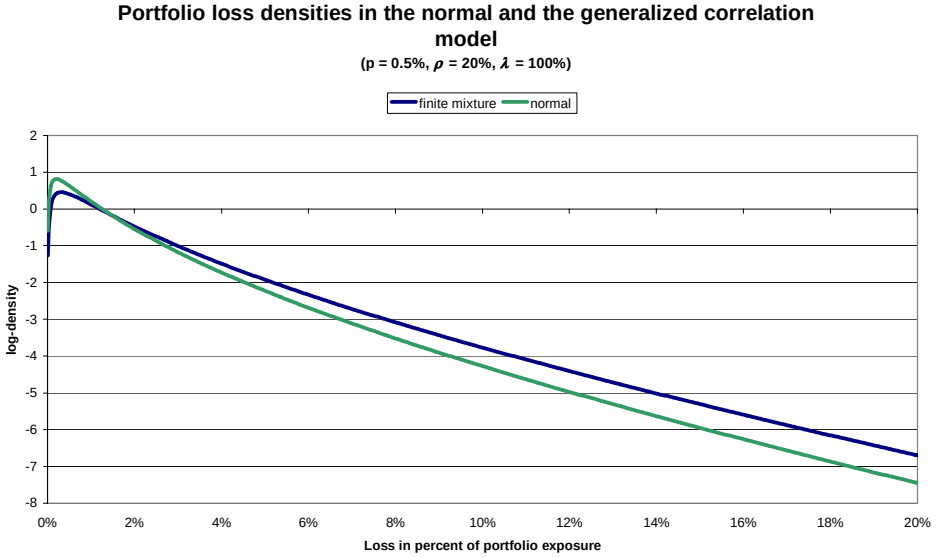


Figure 50: Portfolio densities in the normal and the generalized correlation model

Theorem 7 implies that

$$\frac{L_{d,G\backslash N}(l;p,\rho,\lambda)}{L_{d,N}(l;p,\rho,\lambda)} = \mathbf{E}_W\left(\frac{\varphi\left(\frac{F^{-1}(p)-\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right)}{\varphi\left(\frac{\Phi^{-1}(p)-\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{\rho}}\right)}\right)$$

$$= \mathbf{E}_W\left(\exp\left(-\frac{1}{2}\cdot\left(\frac{F^{-1}(p)-\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{w}\cdot\sqrt{\rho}}\right)^2+\frac{1}{2}\cdot\left(\frac{\Phi^{-1}(p)-\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)}{\sqrt{\rho}}\right)^2\right)\right)$$

$$= \mathbf{E}_W\left(\exp\left(\frac{1}{2w\rho}\cdot\left(\underbrace{w\left[\Phi^{-1}(p)\right]^2-\left[F^{-1}(p)\right]^2}_{=:T_1}\right.\right.\right.$$

$$\left.\left.\left.+\underbrace{2\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)\cdot F^{-1}(p)-2w\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)\cdot\Phi^{-1}(p)}_{=:T_2}\right)\right)\right)$$

The statement $L_{d,N}(l;p,\rho,\lambda) < L_{d,G\backslash N}(xl;p,\rho,\lambda)$ is equivalent to $\frac{L_{d,G\backslash N}(l;p,\rho,\lambda)}{L_{d,N}(l;p,\rho,\lambda)} > 1$.

Note that the term T_1 is constant for given *w*.

$$T_2 = 2\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)\cdot F^{-1}(p) - 2w\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)\cdot\Phi^{-1}(p)$$
$$= 2\sqrt{w}\cdot\sqrt{1-\rho}\cdot\Phi^{-1}(l/\lambda)\cdot\left(F^{-1}(p) - \sqrt{w}\cdot\Phi^{-1}(p)\right)$$

Since $\Phi^{-1}(l/\lambda)\to\infty$ for $l\to\lambda$ and $\exp(-x)\to 0$ for $x\to\infty$, the theorem holds if

$$\mathbf{P}\left\{F^{-1}(p) - \sqrt{w}\cdot\Phi^{-1}(p) > 0\right\} = \mathbf{P}\left\{F^{-1}(p) > \sqrt{w}\cdot\Phi^{-1}(p)\right\} > 0\,. \qquad (*)$$

Let $S := S_w = [w_{\min}; w_{\max}]$ be the support of the mixing distribution *W*.

Let $p < ½$. If $w_{\max} = \infty$, (*) holds trivially because $\Phi^{-1}(p) < 0$.

If $w_{\max} < \infty$, it follows from the continuity and symmetry of *F* that

$$F^{-1}(p) = \inf\left\{l<0 : \int_{\mathbf{R}^{>0}}\int_0^l \frac{1}{\sqrt{w}}\varphi\left(\frac{z}{\sqrt{w}}\right)dz\,W(dw) = p\right\}$$
$$= \inf\left\{l<0 : \int_{\mathbf{R}^{>0}}\Phi\left(\frac{l}{\sqrt{w}}\right)W(dw) = p\right\}$$
$$> \inf\left\{l<0 : \Phi\left(\frac{l}{\sqrt{w_{\max}}}\right) = p\right\} =: \tilde{l} = \sqrt{w_{\max}}\cdot\Phi^{-1}(p)$$

since the mixing distribution *W* is non-degenerate by assumption and $l < 0$.

Hence, there exists $\tilde{w}\in S$ such that $\mathbf{P}\left\{w_{\max} \ge w \ge \tilde{w}\right\} > 0$ and

$$\mathbf{P}\left\{F^{-1}(p) > \sqrt{w}\cdot\Phi^{-1}(p)\right\} > 0\,.$$

The case $p > ½$ can be solved with an analogous argument.

□

It is worth noting that Theorem 9 also proves that the property of tail dependence, that some multivariate distributions show, is not relevant for a specific correlation model to detect higher portfolio risk at high percentiles than the normal correlation model[239].

Tail dependence is an asymptotic measure of dependence of bivariate distributions that is often used to describe dependence of extremal events.

[239] This corrects Nyfeler 2000, p. 50ff., Frey and McNeil 2001, p. 16, and Frey et al. 2001, p. 5ff., who found in simulation experiments that the multivariate t-distribution as risk index distribution led to higher portfolio risk than the normal distribution and claimed this observation to the tail dependence property of the t-distribution.

Definition:[240]

Let X and Y be random variables with distribution functions F_1 and F_2. The coefficient of upper tail dependence of X and Y is

$$\lim_{\alpha \to 1-} \mathbf{P}\left\{Y > F_2^{-1}(\alpha) \mid X > F_1^{-1}(\alpha)\right\} = \lambda$$

provided a limit $\lambda \in [0, 1]$ exists. If $\lambda \in (0, 1]$, X and Y are said to be asymptotically dependent in the upper tail. If $\lambda = 0$, X and Y are said to be asymptotically independent.

Theorem 9 states that the normal correlation model is risk minimal among all correlation models with elliptical risk index distributions. It can, however, be shown that some multivariate elliptical distributions are tail independent as, for instance, the logistic distributions and the symmetric hyperbolic distributions that also include the NIG[241].

(4) More complex types of homogenous portfolios

In order to be able to use the comparatively simple and computationally easy to handle loss distributions of homogenous portfolios as a proxy for the loss distributions of nearly homogenous real world portfolios it would be desirable to relax some of the structural assumptions on the homogeneity of the portfolios made in the theorems above. Indeed, even retail portfolios, where exposure concentrations on individual addresses are rare and sectorial distinctions of counterparties are negligible, usually do contain clients with heterogeneous probabilities of default.

In this section, we, therefore, want to present analytic and semi-analytic solutions for homogenous portfolios that allow for clients to have different default probabilities.

With the notation of the previous sections, we define an 'almost homogenous portfolio' H as the union of sub-portfolios H_j, $j = 1,...,h$,

$$H = \bigcup_{j=1}^{h} H_j ,$$

where the sub-portfolios H_j are homogenous with exposures E, risk index correlations[242] ρ, probabilities of default p_j and expected losses given default λ_j for $j = 1,...,h$. We assume that all clients in H have the same risk index correlations ρ and exposures E. Types of clients

[240] See Schmidt (2002), p. 302, definition 1.1 or Embrechts et al. (1999), p. 18, definition 7.

[241] See Schmidt (2002), p. 324f., theorem 6.6 and theorem 6.8.

[242] See above footnote 209.

differ only by default probability p_j and expected loss given default λ_j. Moreover, each type of client is represented by a homogenous sub-portfolio H_j containing n_j clients for $j = 1,...,h$.

Without loss of generality, we may assume that all clients' risk index distributions are centered and have variance 1.

Let *D* be the distribution of risk indices. We assume that *D* is a mixture of normals with mixing distribution *W*, such that

$$\sqrt{w} \cdot X \sim D$$

if

$$w \sim W \text{ and } X \sim \mathsf{N}(0,1).$$

In the generalized correlation model, in an almost homogenous portfolio each client's risk index X_i is then given as

$$X_i = \sqrt{w} \cdot \left(\sqrt{\rho} \cdot Y + \sqrt{1-\rho} \cdot Z_i \right)$$

with $i = 1,...,N$ if $N := \sum_{j=1}^{h} n_j$ is the number of clients in portfolio *H* where *Y* and Z_i are standard normally distributed. Note that all clients in *H* depend on the same single systematic risk factor *Y*, even if they are in different sub-portfolios.

For ease of exposition, we assume in the following that $n_j = n$, for $j = 1,...,h$.

Generalizing Theorem 2 we derive an analytic solution for the asymptotic inverse portfolio loss distribution in the normal correlation model, i.e. for the case $w \equiv 1$.

Theorem 10:
In the normal correlation model, in an almost homogenous portfolio containing *n* clients each having an exposure of $E = 1/n$ and risk index correlations of $0 \le \rho < 1$ the α-percentile of the asymptotic portfolio loss distribution is given as

$$\lim_{n\to\infty} L^{-1}(\alpha; p_1,..., p_h, E(n), \rho, \lambda_1,..., \lambda_h) = L^{-1}(\alpha; p_1,..., p_h, \rho, \lambda_1,..., \lambda_h)$$
$$= \sum_{j=1}^{h} \lambda_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho} \cdot \Phi^{-1}(1-\alpha)}{\sqrt{1-\rho}} \right)$$

Proof:

We closely follow the lines of the proof of Theorem 2. By definition of the normal correlation model, client *i* in sub-portfolio H_j defaults if

$$X_i = \sqrt{\rho} \cdot Y + \sqrt{1-\rho} \cdot Z_i \leq \Phi^{-1}(p_j)$$
$$\Leftrightarrow Z_i \leq \frac{\Phi^{-1}(p_j) - \sqrt{\rho} \cdot Y}{\sqrt{1-\rho}}$$

for $i = 1,...,n$ and $j = 1,...,h$.

Hence, client *i*'s probability of default conditional to *Y* is given as

$$\mathbf{P}\{\text{client } i \text{ defaults} \mid Y\} = \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho} \cdot Y}{\sqrt{1-\rho}}\right)$$

because Z_i is standard normally distributed for $i = 1,...,n$.

Moreover, since the idiosyncratic components Z_i of clients' risk indices are stochastically independent, it follows from the law of large numbers that the percentage of clients defaulting in each sub-portfolio given *Y* is equal to their conditional probability of default almost surely one if $n \to \infty$.

Asymptotically the number of defaulting clients in each sub-portfolio goes to infinity as well, if the conditional default probability is positive. Thus, again by the law of large numbers, the loss conditional to *Y* in sub-portfolio H_j is equal to

$$\text{Loss in } H_j \mid Y = \lambda_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho} \cdot Y}{\sqrt{1-\rho}}\right)$$

since the individual loss given default rates in H_j are stochastically independent and limited with the same mean λ_j for $j = 1,...,h$.

The loss in the entire portfolio *H* conditional to *Y* is then given as

$$\text{Loss in } H \mid Y = \sum_{j=1}^{h} (\text{Loss in } H_j \mid Y) = \sum_{j=1}^{h} \lambda_j \cdot \Phi\left(\frac{\Phi^{-1}(p_j) - \sqrt{\rho} \cdot Y}{\sqrt{1-\rho}}\right)$$

because all clients in *H* depend exclusively on the same systematic risk factor *Y*.

The inverse cumulative loss distribution function can again immediately be derived from this expression for portfolio losses because the systematic risk factor *Y* is the only remaining ran-

dom component in portfolio losses and the conditional portfolio loss given Y is monotonously decreasing in Y. The α-percentile of the portfolio loss distribution, therefore, maps one-to-one to the $(1-\alpha)$-percentile of the distribution of Y. The $(1-\alpha)$-percentile of Y is given by $\Phi^{-1}(1-\alpha)$ because Y is standard normally distributed. This completes the proof.

□

Thus, in the normal correlation model, in almost homogenous portfolios the α-percentile of the loss distribution of the entire portfolio is just the sum of the α-percentiles of the loss distribution of the sub-portfolios. This means that the portfolio value at risk, which is a percentile of the portfolio loss distribution for a specific value of α, can be calculated separately for all sub-portfolios and then be aggregated over the sub-portfolios by simple addition.

Moreover, the calculation of the percentile function only requires the solving of the normal cumulative distribution function, a feature that is provided by virtually all spreadsheet calculators such as Lotus 1-2-3 and Microsoft Excel.

This makes the percentile function of almost homogenous portfolios an excellent basis for the quick, easy, and approximate, but also methodologically precise calculation of a bank's portfolio credit risk as it is, for instance, essential for the computation of regulatory capital requirements. See Gersbach/Wehrspohn (2001) on how the impact of heterogeneous exposures and correlation and diversification effects between loosely connected sectors can be incorporated into the percentile function by an adjusted correlation assumption and a modified aggregation rule for sub-portfolio risks.

This analytic solution of the percentile function of the portfolio loss distribution in almost homogenous portfolios was implemented by the author in the 'New Basel Capital Accord Calculator – January 2001' which was published by Computer Sciences Corporation on the website http://www.creditsmartrisk.com.

In the generalized correlation model for non-constant mixing distributions W there is no closed form analytic solution of the percentile function of the loss distribution of almost homogenous portfolios. However, one can show analogously to Theorem 10 and Theorem 5 that the loss of portfolio H conditional to the systematic factors w and Y is given as

$$\text{Loss in } H \mid w, Y = \sum_{j=1}^{h} \left(\text{Loss in } H_j \mid w, Y \right) = \sum_{j=1}^{h} \lambda_j \cdot \Phi\left(\frac{F^{-1}(p_j) - \sqrt{w} \cdot \sqrt{\rho} \cdot Y}{\sqrt{w} \cdot \sqrt{1-\rho}} \right).$$

To derive the portfolio loss distribution it is, therefore, sufficient to simulate portfolio losses conditional to *w* and *Y* by drawing independent random numbers from the mixing distribution *W* and the standard normal distribution (see Figure 51).

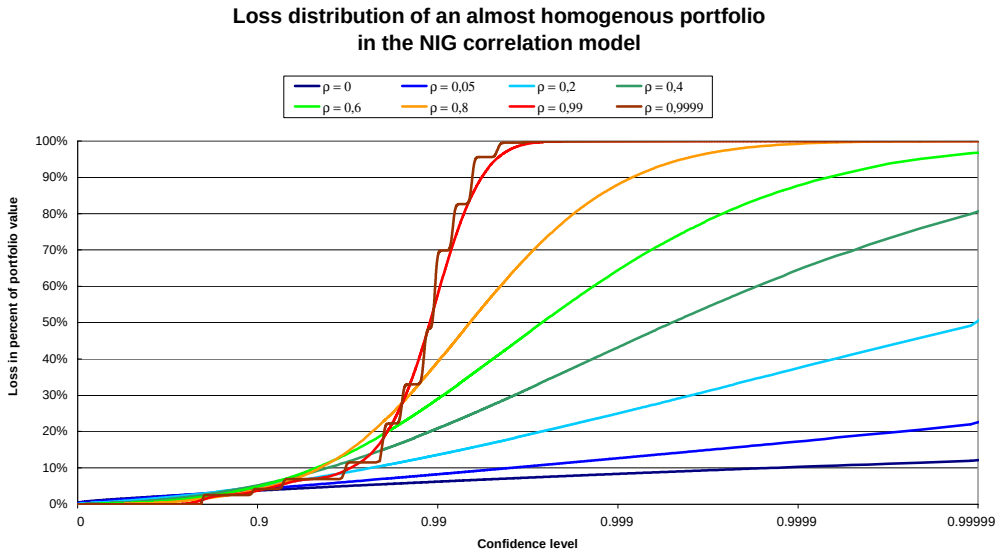


Figure 51: Loss distribution of an almost homogenous portfolio in the NIG correlation model[243]

(5) Speed of convergence

The percentile and cumulative distribution functions in the normal and generalized correlation model discussed so far are asymptotic results that hold if the number of clients in the portfolio goes to infinity. This is, however, a prerequisite that can hardly be fulfilled by real world portfolios even if they are approximately homogenous such as retail and credit card portfolios. We therefore, ask the question how quickly the loss distribution of a finite homogenous portfolio, a homogenous portfolio containing only finitely many clients, converges in the number of clients against the asymptotic loss distribution.

For this purpose, we derive an analytic solution of the loss distribution of finite homogenous portfolios in the generalized correlation model.

---

[243] The portfolio was defined by the following characteristics:

Sub-portfolio	Prob. of default	Exposure
I	0,33%	0,48%
II	0,47%	3,96%
III	0,65%	12,94%
IV	0,83%	12,79%
V	1,04%	21,55%
VI	1,20%	15,21%
VII	1,59%	10,86%
VIII	2,02%	10,71%
IX	3,30%	4,56%
X	7,25%	2,73%
XI	10,68%	1,60%
XII	20,20%	2,61%

All loss given default rates were set to 100%. The parameters of the NIG were chosen as $\alpha = \delta = 3$.

Theorem 11:

In the generalized correlation model, in a homogenous portfolio containing *n* clients each having an exposure of $E = 1/n$ and risk index correlations $0 \le \rho < 1$ the portfolio loss distribution is given as

$$L_n\left(\frac{k}{n}; p, 1/n, \rho\right) = \mathbf{P}\left\{\text{Loss} \le \frac{k}{n}\right\}$$

$$= \sum_{m=0}^{k} \binom{n}{m} \cdot \int_{\mathbf{R}^+} \int_{\mathbf{R}} \left[\Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)\right]^m \cdot \left[1 - \Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)\right]^{n-m} \cdot \varphi(Y)dY \cdot W(dw)$$

where $\varphi(\cdot)$ is the standard normal density function and $k \in \{0,..., n\}$ is the number of defaulting clients in the portfolio.

Proof:

In the generalized correlation model, in a homogenous portfolio client *i* defaults if

$$X_i = \sqrt{w}\cdot\sqrt{\rho}\cdot Y + \sqrt{w}\cdot\sqrt{1-\rho}\cdot Z_i \le F^{-1}(p)$$

$$\Leftrightarrow Z_i \le \frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}$$

for $i = 1,..., n$.

Hence, client *i*'s probability of default conditional to *Y* and *w* is given as

$$\mathbf{P}\{\text{client } i \text{ defaults} \mid w, Y\} = \Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)$$

because Z_i is standard normally distributed for $i = 1,..., n$.

The conditional probability given *w* and *Y* that *k* of *n* clients default in the portfolio is then simply defined by the binomial distribution

$$\mathbf{P}\left\{\text{Loss} = \frac{k}{n} \mid w, Y\right\} = \binom{n}{k} \cdot \left[\Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)\right]^k \cdot \left[1 - \Phi\left(\frac{F^{-1}(p) - \sqrt{w}\cdot\sqrt{\rho}\cdot Y}{\sqrt{w}\cdot\sqrt{1-\rho}}\right)\right]^{n-k}$$

because clients are conditionally independent by assumption given *w* and *Y*.

This directly implies the loss distribution as stated in the theorem.

□

Figure 52 shows the loss distributions from Theorem 11 for various numbers of clients and the asymptotic loss distributions for the normal and the NIG correlation model.

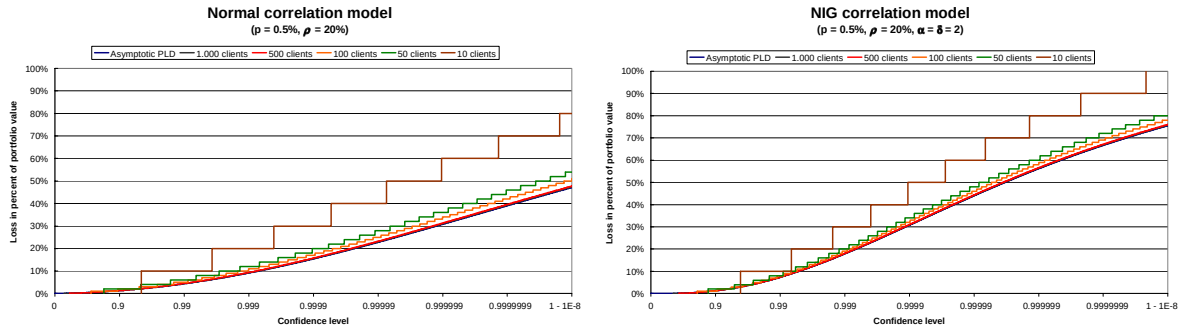


Figure 52: The speed of convergence towards the asymptotic portfolio loss distribution in the generalized correlation model

It can be seen well that an increasing number of clients in the portfolio increases diversification and reduces risk at all confidence levels. However, 500 clients seem to be sufficient to fully exploit diversification effects and to approximately reach the asymptotic portfolio loss distribution.

As a consequence, in many real world portfolios the lack of size in terms of numbers of clients cannot be the reason why analytic loss distributions for homogenous portfolios should not be used for the calculation of portfolio risk. It is rather the lack of homogeneity within the portfolio. Note, however, that all results discussed so far can be generalized to portfolios consisting only of homogenous segments if the (heterogeneous) dependencies between the segments can be simulated. The over-all loss distribution is then merely a mixture of the analytic loss distributions of the homogenous sub-portfolios. The previous section on almost homogenous portfolios gave a simple example of how such a generalization could look like.

b) Estimation of risk index distributions

As we have shown in the previous sections, the choice of the risk index distribution significantly influences the resulting loss distribution of a given portfolio. To control this type of model risk it is, therefore, essential to employ a quantitative technique to estimate the risk index distributions in dependence on the available risk index data.

A general approach to the estimation of risk index distributions could, for instance, make use of non-parametric kernel smoothers. For simplicity, however, we supply a parametric estima-

tion technique[244] under the assumption that the actual risk index distribution is an element of the family of symmetric centered normal inverse Gaussian distributions.

Let d be the number of risk indices in the portfolio. For observations $i = 1,...,n$, let $x_i = (x_{1i},...,x_{mi})$ be a m-dimensional vector of realized values of these risk indices. The m-dimensional symmetric centered NIG has the density

$$nig(x; m,\alpha,\delta,\mu,\Delta) = 2\delta\left(\frac{\alpha}{2\pi}\right)^{\frac{m+1}{2}} \exp(\alpha\delta)\frac{\mathbf{K}_{(\delta+1)/2}\left(\alpha\sqrt{\delta^2 + x'\Delta^{-1}x}\right)}{\left(\delta^2 + x'\Delta^{-1}x\right)^{(m+1)/4}}$$

where $\mathbf{K}_{(\delta+1)/2}$ is the modified Bessel function of the third kind, $\alpha,\delta > 0$, $\mu \in \mathbf{R}^m$ and $\Delta \in \mathbf{R}^{m\times m}$ is a positive definite matrix with determinant $|\Delta| = 1$. Expectation and variance of $X \sim nig(m,\alpha,\delta,\mu,\Delta)$ can be shown to be given as

$$\mathbf{E}X = \mu$$

and

$$\mathbf{Var}(X) = \Sigma := \frac{\delta}{\alpha}\Delta,$$

i.e. Δ is identical to the distribution's covariance matrix Σ normed to have a determinant of 1:

$$\Delta = \frac{\Sigma}{|\Sigma|^{1/m}}$$

The estimation of the unknown parameters α,δ,μ,Δ proceeds in two steps. The first step follows a method of moments approach[245], i.e. we estimate the sample mean $\hat{\mu} \in \mathbf{R}^m$ and the sample covariance matrix $\hat{\Sigma} \in \mathbf{R}^{m\times m}$ using standard techniques and get an estimate for Δ as

$$\hat{\Delta} = \frac{\hat{\Sigma}}{|\hat{\Sigma}|^{1/m}}.$$

To derive the remaining parameters α and δ, a maximum likelihood approach can be used. Standardizing the observations $x_i \in \mathbf{R}^m$, $i = 1,...,n$, to $y_i := x_i' \Delta^{-1} x_i$ the log-likelihood function has the form

[244] This approach was developed by Eberlein and Prause 1998 and Prause 1999.

[245] Cf. Prause 1999, p. 108.

$$\mathrm{L}(x;\alpha,\delta)=n\left[\ln(2\delta)+\alpha\delta+\frac{m+1}{2}\ln\left(\frac{\alpha}{2\pi}\right)\right]+\sum_{i=1}^{n}\left[\ln\mathbf{K}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)-\frac{m+1}{4}\ln\left(\delta^2+y_i\right)\right]$$

The log-likelihood function can be directly maximized using numerical techniques or first derivatives can be calculated and set to zero.

With the abbreviation

$$\mathbf{R}_\lambda(x):=\frac{\mathbf{K}_{\lambda+1}(x)}{\mathbf{K}_\lambda(x)}$$

and

$$\left(\ln\mathbf{K}_\lambda(x)\right)'=\frac{\lambda}{x}-\mathbf{R}_\lambda(x)$$

the first derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{L}=n\left[\delta+\frac{m+1}{2}\right]-\sum_{i=1}^{n}\sqrt{\delta^2+y_i}\,\mathbf{R}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\mathrm{L}=n\left[\delta^{-1}+\alpha\right]-\sum_{i=1}^{n}\frac{\alpha\delta}{\sqrt{\delta^2+y_i}}\mathbf{R}_{(d+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right).$$

To verify that it is indeed a maximum which has been achieved and not some other stationary point, second-order derivatives have to be examined.

With

$$\mathbf{S}_\lambda(x):=\frac{\mathbf{K}_{\lambda+2}(x)\mathbf{K}_\lambda(x)-\mathbf{K}_{\lambda+1}^2(x)}{\mathbf{K}_\lambda^2(x)}$$

and

$$\left(\ln\mathbf{K}_\lambda(x)\right)''=\mathbf{S}_\lambda(x)-\frac{\mathbf{R}_\lambda(x)}{x}-\frac{\lambda}{x^2}$$

they are given as

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha^2}\mathrm{L}=-\frac{n(d+1)}{\alpha^2}-\sum_{i=1}^{n}\left(\delta^2+y_i\right)\left[\frac{\mathbf{R}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)}{\alpha\sqrt{\delta^2+y_i}}-\mathbf{S}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)\right]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\alpha\,\mathrm{d}\delta}\mathrm{L} = n - \sum_{i=1}^{n}\left[\frac{2\delta\,\mathbf{R}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)}{\alpha\sqrt{\delta^2+y_i}} - \alpha\delta\,\mathbf{S}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)\right]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\delta^2}\mathrm{L} = -\frac{n}{\delta^2} + \alpha^2\delta^2\sum_{i=1}^{n}\left(\delta^2+y_i\right)^{-1}\mathbf{S}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)$$
$$-\sum_{i=1}^{n}\mathbf{R}_{(m+1)/2}\left(\alpha\sqrt{\delta^2+y_i}\right)\left(\left(\delta^2+y_i\right)^{-1/2} + 2\delta^2\left(\delta^2+y_i\right)^{-3/2}\right)$$

c) Copulas

A possibility to further generalize the correlation model beyond elliptic risk index distributions is not only to specify a model by one-dimensional margins of risk index distributions and their linear correlations, but to employ copula functions[246] to directly describe the multivariate distribution of risk indices.

All information about the individual behavior and the dependence of real-valued random variables $X_1,...,X_n$ is fully contained in their joint distribution function

$$\mathrm{F}(x_1,...,x_n) = \mathbf{P}\{X_1 \le x_1,...,X_n \le x_n\}$$

The objective to separate the joint distribution function into a part that expresses the marginal distributions and another part that defines their dependence structure has led to the concept of copula functions.

Definition:

A copula is the distribution function of a random vector in $\mathbf{R}^n$ with uniform-(0, 1) marginals.

Let $X = (X_1,...,X_n) \sim F$ with continuous marginal distributions $F_1,...,F_n$. Then the copula C of X is given as the joint distribution of $(F_1(X_1),...,F_n(X_n))$ because $F_i(X_i) \sim U(0,1)$ for $i = 1,...,n$ due to the percentile transformation:

$$\begin{aligned} F(x_1,...,x_n) &= \mathbf{P}\{X_1 \le x_1,...,X_n \le x_n\} \\ &= \mathbf{P}\{F_1(X_1) \le F_1(x_1),...,F_n(X_n) \le F_n(x_n)\} \\ &= C(F_1(x_1),...,F_n(x_n)) \end{aligned}$$

The copula C of F is, thus, given as

[246] For details on copula functions confer to Nelson 1999, Embrechts et al. 1999, or Embrechts et al. 2001.

$$C(u_1,...,u_n) = F\left(F_1^{-1}(u_1),...,F_n^{-1}(u_n)\right).$$

If the joint distribution function F has continuous margins then the copula function C is uniquely determined. It is intuitive to consider C as the dependence structure of F because C is invariant under increasing continuous transformations of the marginals $F_1,...,F_n$.

Theorem 12[247]*:*

If $(X_1,...,X_n)$ has copula C and $T_1,...,T_n$ are increasing continuous functions, then $(T_1(X_1),..., T_n(X_n))$ also has copula C.

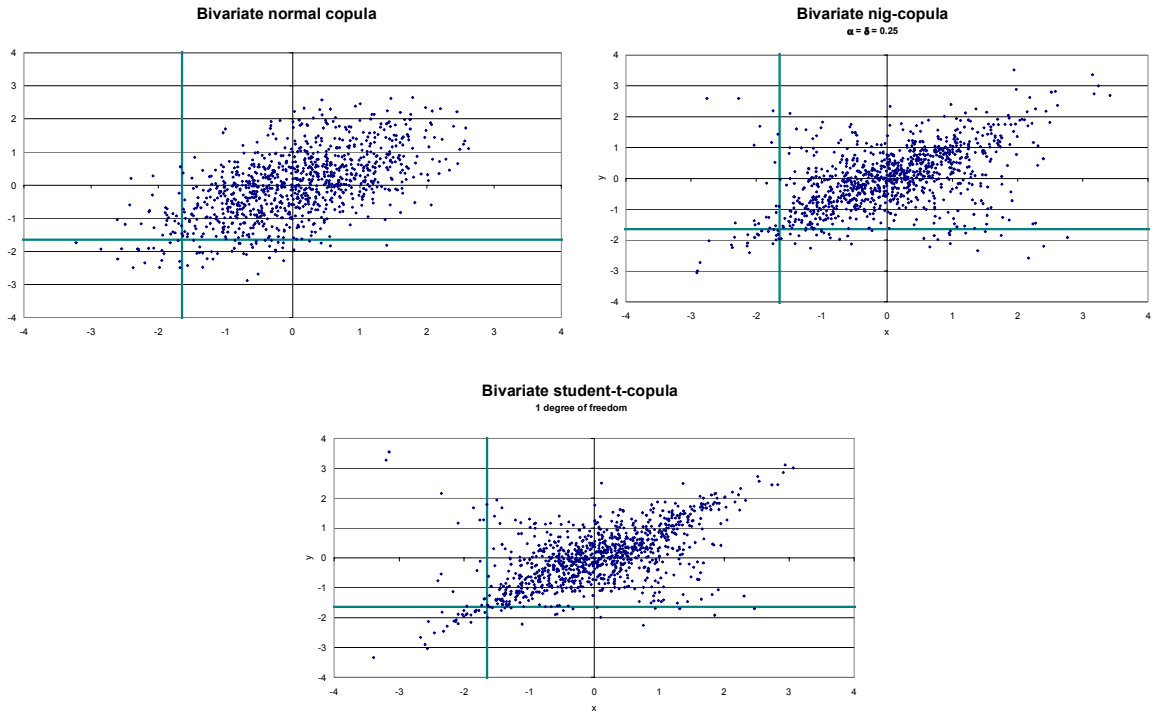


Figure 53: Bivariate distributions with standard normal marginals and different copulas

Figure 53 illustrates the effect of the copula function on the shape of the joint distribution.[248] It clearly shows that the bivariate joint distribution is not uniquely determined by the marginals and their correlation. Note that only the bivariate normal distribution is elliptic while both the NIG- and the t-copula induce a star-shaped joint distribution.

To demonstrate the effect of the copula on clients' joint default probability if the joint distribution is interpreted as the distribution of risk indices in analogy to the correlation model. The 5% default threshold is indicated in the graphs. Of the 1,000 sample points in each chart, 19 imply a joint default in the normal-correlation-model, 23 in the chosen NIG-correlation-model, and 26 in the t-copula-correlation-model.

247 Cf. Embrechts et al. 2001, p. 5, theorem 1.6.

248 All marginal distributions are standard normal with correlation ρ = 0.6.

Although being a very desirable generalization of the correlation model described in the previous sections, the copula-correlation-model as we might call it, has two major drawbacks which presently diminish its practical use significantly: firstly, the estimation of a high-dimensional copula is a serious problem, especially if the available data is scarce[249]. Secondly, the concept of (linear) correlation, so inherent to the correlation model, is not entirely defined on the level of the copula function, but also depends on the chosen marginals. For this reason, it is sometimes difficult to find the correct parameterization of the copula to get the desired risk index correlations. At this point, additional research has to be done to possibly replace the concept of correlation by a more suitable and easy to handle measure of dependence.

3. Random default probabilities (Credit Risk+ and Credit Portfolio View)

A more direct concept of dependence than in the correlation model is found in Credit Risk+ and Credit Portfolio View. Here it is explicitly assumed that clients' default probabilities are not constant, but vary as a function of some random systematic risk factors, and that clients are independent conditional to the systematic factors.

a) Credit Risk+

In Credit Risk+ systematic risk factors are modeled as hidden variables that induce clients' default probabilities to be gamma distributed with a given mean and variance[250]. In order to be able to compute the portfolio loss distribution analytically, the authors of the model assume that systematic risk factors only refer to the clients in a specific sector while risk factors of different sectors are independent by supposition. Note that this presumption implies that clients in different sectors are independent as well, a problematic concealed structural decision for a portfolio model. Only clients who are at least partially represented in the same industrial sectors appear to be dependent in their default behavior[251].

Also for the sake of analytical solubility, the architects of the model sort clients into relatively homogenous groups where all counterparties have the same default probability and volatility and similar exposures. They also assume the number of clients defaulting in each group to be Poisson distributed if a certain default probability is given.

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249 See Embrechts et al. 1999, p. 3, for references.

250 See above section I.B.5 and CSFP 1996.

251 Hence, in order to have a certain basic dependence between all, say, corporate clients, a common sector would have to be created where all corporate clients are represented. However, if such an approach is used, the definition of sectors in the model would deviate from firms' industrial sectors and become somewhat artificial and difficult to quantify.

Note that the Poisson distribution - a limit distribution of the in this context exact binomial distribution of the number of defaults - takes on each positive integer with positive probability. Thus, here again an implicit assumption is made, that is that each homogenous group contains an infinite number of clients.

Although this supposition has little importance for large relatively homogenous portfolios such as retail or credit card portfolios, it might effectively spoil the risk calculations for corporate and junk bond portfolios or the respective sub-portfolios in a larger set of exposures. Corporate portfolios usually contain extremely heterogeneous exposures with often less than 3% of clients holding more than 50% of total portfolio exposure. Especially the number of very large exposures is generally particularly small so that the size and the probability of portfolio losses can easily be overestimated in this segment.

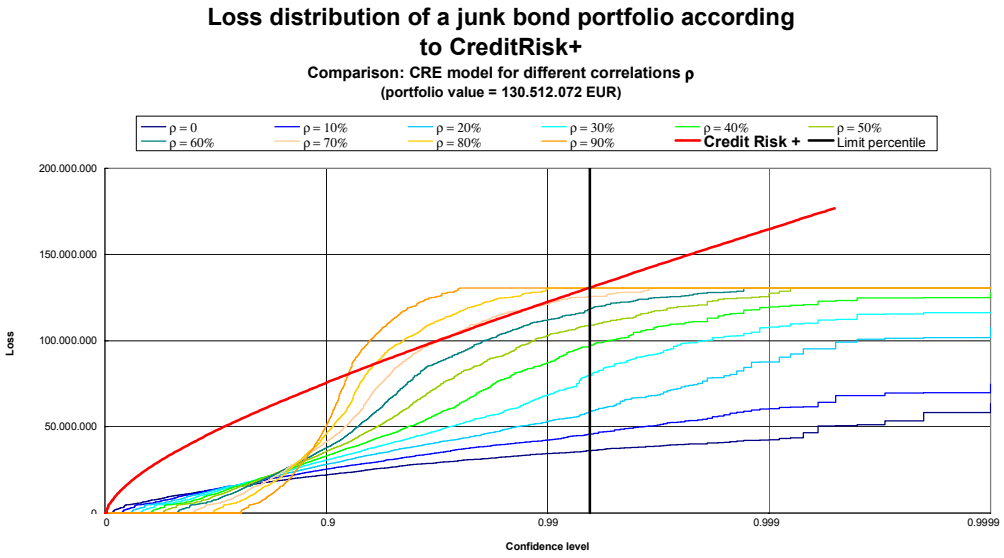

Figure 54: Loss distribution of a junk bond portfolio according to Credit Risk+

A similar effect occurs in junk bond portfolios. In this context, Credit Risk+ tends to find loss percentiles at high confidence levels that exceed total portfolio exposure and, thus, leads to systematic conceptual errors. Figure 54 gives an example for this effect. It is worth commenting that the calculations of the Credit Risk+ loss distribution were performed for a portfolio and with a software that both were supplied by Credit Suisse Financial Products, the developers of the Credit Risk+ model. As a benchmark, portfolio loss distributions were also computed for the same portfolio with the CRE model for various levels of risk index correlations.

Two things are striking in Figure 54. Firstly, for this portfolio, in Credit Risk+ portfolio losses surpass total portfolio exposure at confidence levels above 99.35% while in the CRE model

the maximum portfolio loss is always given by total portfolio exposure as it should be in a framework where the portfolio exposure is not subject to risk itself.

Secondly, the portfolio loss distribution produced by Credit Risk+ is much more continuous than the portfolio loss distribution resulting from the CRE model. This effect is again a consequence of the assumed Poisson approximation in Credit Risk+ because it implies that not only the large, but also the small individual exposures in the portfolio can default in an arbitrarily large number. Hence, the big jumps in the portfolio loss distribution caused by the default of a large exposure will be filled up.

It is worth noting, though, that these conceptual pitfalls in Credit Risk+ have their origin in a number of decisions that were exclusively made for analytical tractability of the model. All of them could easily be avoided if computer simulations were allowed to solve the model.

b) Credit Portfolio View

As already described in section I.B.6.b) above, in Credit Portfolio View a segment's conditional probability of default[252] given a macroeconomic situation and a random shock is

$$p \mid X, \varepsilon = \frac{1}{1+\exp(\theta X + \tau\varepsilon)}$$

where X is a vector of macroeconomic factors, ε a standard normal random innovation, θ a parameter vector and τ a scalar parameter.

4. A brief reference to the literature and comparison of the models applied to homogenous portfolios

The discussion in the previous sections has shown that there are numerous differences between the leading credit portfolio models in terms of relevant input factors and their estimation and of modeling assumptions. Also sometimes conceptual errors and imprecisions may occur in one model, but not in the others. As a consequence, it is highly likely that the models will lead to very different results even if they are applied to the same portfolio.

However, there is a major current in the literature[253] that intends to place the models within the same mathematical framework and tries to show that the models lead to practically equivalent results, if the input parameters are harmonized properly implying "that relative

[252] At this point, we ignore the unpublished derivation of conditional transition probabilities for different rating grades and industrial sectors and consider all speculative grade companies in the same sector as one segment.

[253] See for instance Koyluoglu and Hickman 1998 and Gordy 2000.

'theoretical correctness' need not rank among a user's model selection criteria, which might then consist primarily of practical concerns…"[254].

Koyluoglu and Hickman find a common mathematical framework in the fact that clients' defaults are conditionally independent in all models given the respective set of systematic risk factors. This conceptual phenomenon is a general feature of all structural risk models, though, not only of credit risk models, that does not give any hint concerning a model's 'theoretical correctness'.

What is more, calculation results can sometimes be subject to interpretation errors. For instance, Koyluoglu and Hickman calculate the distributions of conditional default probabilities given the respective set of systematic risk factors for a standardized type of firm[255] with the normal correlation model, Credit Risk+ and Credit Portfolio View. To compare the resulting distributions, they defined a 'tail-agreement statistics'

$$\Xi_z(f,g) = 1 - \frac{\int_z^\infty |f(x) - g(x)| \, \mathrm{d}x}{\int_z^\infty f(x)\mathrm{d}x - \int_z^\infty g(x)\mathrm{d}x}$$

where f and g are density functions of two distributions of conditional default probabilities to be compared and z defines the beginning of the 'tail'[256]. Koyluoglu and Hickman define the tail as the area more than two standard deviations above the mean of the distributions of conditional default probabilities, i.e. $z = p + 2\sigma$ and get

	Ξ
NCM vs. CPV	**94.90%**
NCM vs. CR+	**93.38%**
CPV vs. CR+	**88.65%**

which they take as an indication for the similarity of the models[257].

To see that this conclusion might be misleading for the standard portfolio statistics such as value at risk and shortfall and be due to the 'tail-agreement statistics' used for the comparison, note that in homogenous portfolios with a total exposure of 1 the distribution of conditional

254 Koyluoglu and Hickman 1998, p. 18.

255 They consider firms with an unconditional default probability p = 1.16% and a standard deviation of the conditional default probability of σ = 0.9%. They chose these parameter values to match Moody's 'All Corporate' default experience for 1970-1995, as reported in Carty and Lieberman 1996. See Koyluoglu and Hickman 1998, p. 12ff.

256 Koyluoglu and Hickman 1998, p. 18.

257 Koyluoglu and Hickman 1998, p. 14f.

default probabilities and the asymptotic portfolio loss distribution in the sense of Theorem 5[258] are identical.

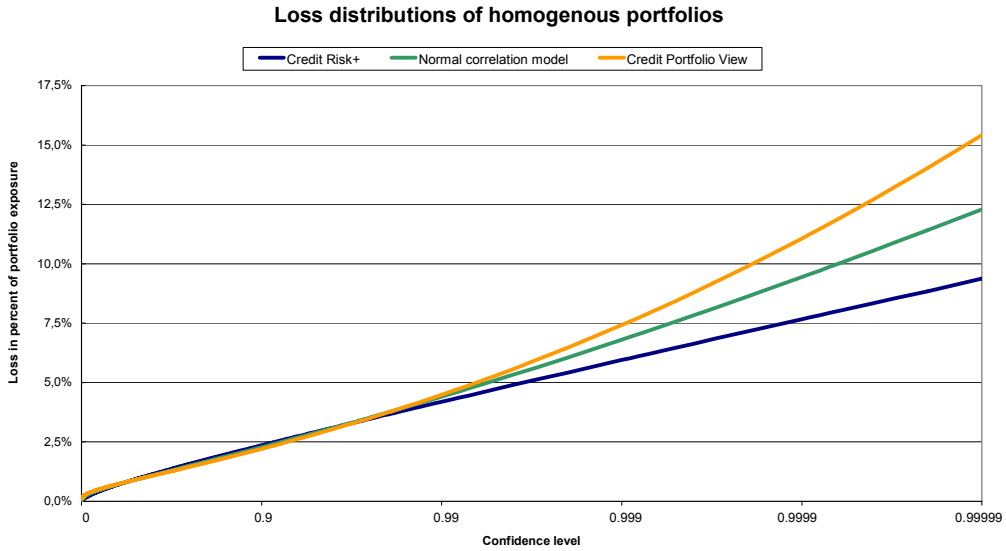

Figure 55: Loss distributions resulting from Credit Risk+, Credit Portfolio View and the normal correlation model

Figure 55 shows the portfolio loss distributions that Koyluoglu and Hickman compared. Contrary to their results, it is particularly striking that the three models under consideration strongly deviate especially for percentiles above the 99% confidence level indicating the substantial structural differences between the models even in this highly standardized case.

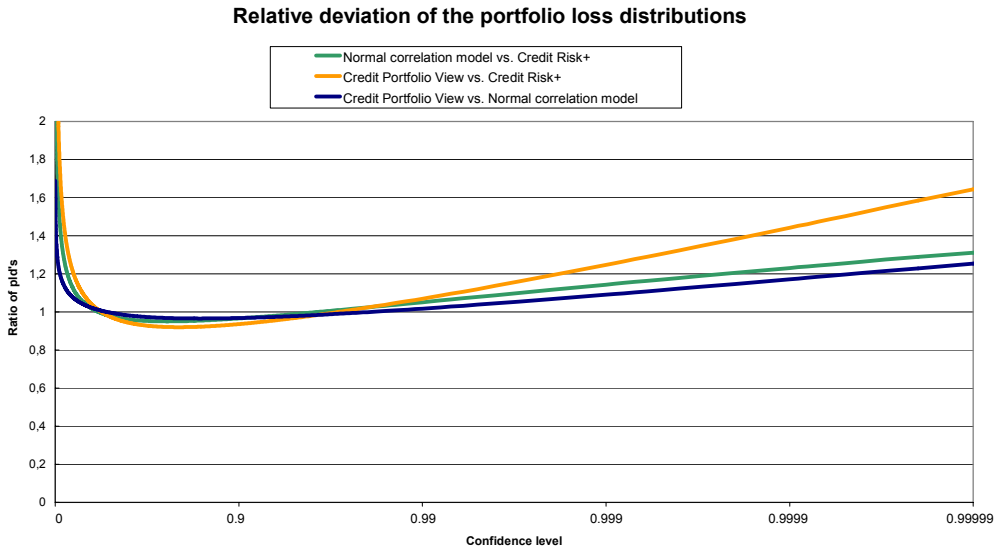

Figure 56: Relative deviation of the portfolio loss distributions

[258] I.e. the loss distribution that results if the number of clients in the portfolio tends to infinity.

At the 99.9% confidence level, that was chosen as the standard measure of credit portfolio risk by the Basel Committee for Banking Supervision[259], we find the following *relative* deviations between the loss distributions (Figure 56):

	99.9%-VaR	99.9%-Shortfall
CPV vs. NCM	9.07%	12.90%
NCM vs. CR+	14.37%	18.43%
CPV vs. CR+	24.73%	33.71%

and, thus, a divergence of from 9% to almost 25% between the 99.9%-values at risk and a 12.9% to 33.7% discrepancy between the 99.9%-shortfalls.

This evidence clearly shows that superficial mathematical similarities between models do not at all imply resemblance from the risk managers point of view.

5. Event driven dependencies (CRE model)

A complementary concept of dependence between clients in credit portfolios is event driven. Examples could be country risk and microeconomic risks between individual counterparties. Event driven dependencies occur if a client's ability to repay a credit or any other obligation is directly affected if a certain incident arises or not[260].

For instance, creditors who are resident abroad can only serve their debts if international money transfers are not interrupted for reasons such as political or economic crises or the local central bank's lack of foreign currencies. The probability of the disruption of a country's international money transfers is typically measured by the countries rating as it is supplied by the international rating agencies such as Moody's, Standard & Poor's and Fitch's.

The other canonical illustration for event driven risks is microeconomic dependency between individual clients. Take for example a firm who has only one customer. An insolvency of its single purchaser would certainly not leave the firm unaffected, but force it to quickly find one or more new clients to avoid default itself.

[259] See BCBS 2001.

[260] For details see also section I.B.7.a)(2) above.

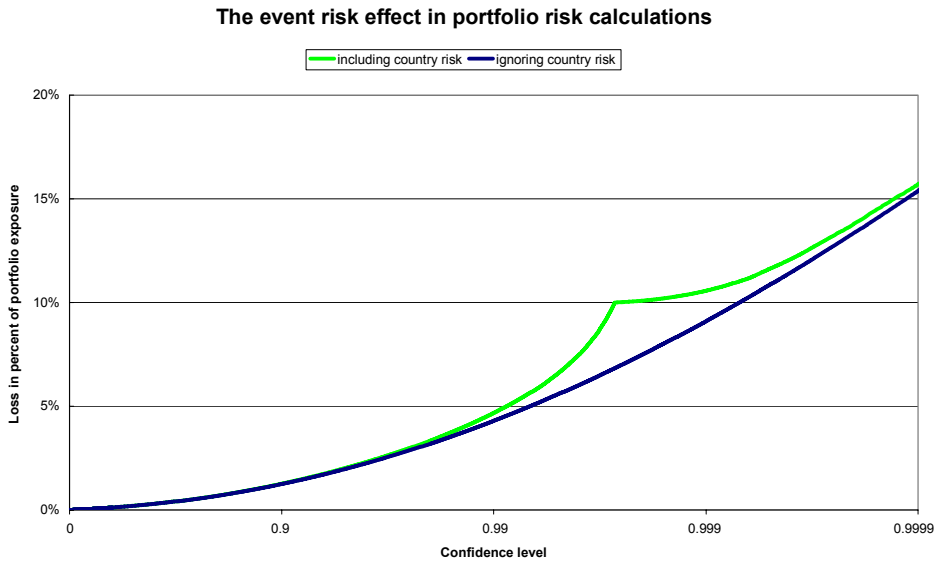

Figure 57: The event risk effect in portfolio risk calculations: country risk

Figure 57 demonstrates the effect of the inclusion of country risk in the calculation of portfolio risk. In the example, 10% of all clients in an otherwise homogenous portfolio[261] lodge abroad. The probability of an interruption of money transfers is set to 0.2%. Due to the homogeneity of the portfolio, the loss distribution is not discontinuous, but steeply rises before 0.2% to incorporate the additional loss caused by country risk.

Opposite to risk index correlations or random default probabilities, it is a characteristic feature of event risks that they can model asymmetric dependencies between portfolio components. In case of country risk an interruption of money transfers between a country and the rest of the world simultaneously affects all clients who lodge in that country. An insolvency of a client in that country, however, usually does not affect the country's ability to transfer money. Although a counterparty's ability to participate in international business decisively depends on the quality of his home country, this is not true vice versa.

In case of microeconomic dependencies between firms, dependencies may be positive in both directions and of different size. Moreover, since clients can affect each other in this setting, the influence of microeconomic dependencies on portfolio risk can be modeled in several rounds leading to a cascading effect. In round 0 portfolio risk is calculated as if no microeconomic dependencies were present in the portfolio finding a certain number of defaults. In round 1 the consequences of these defaults are analyzed. Due to individual dependencies of clients who are not in insolvency so far (in the model) on defaulted business partners some of

[261] Clients' individual default probability is $p = 0.5\%$ with correlations of $\rho = 20\%$ in the normal correlation model.

the still intact clients will be drawn into the crisis and default in turn. Round 2 proceeds similarly and investigates the effect of these new collapses and so on.

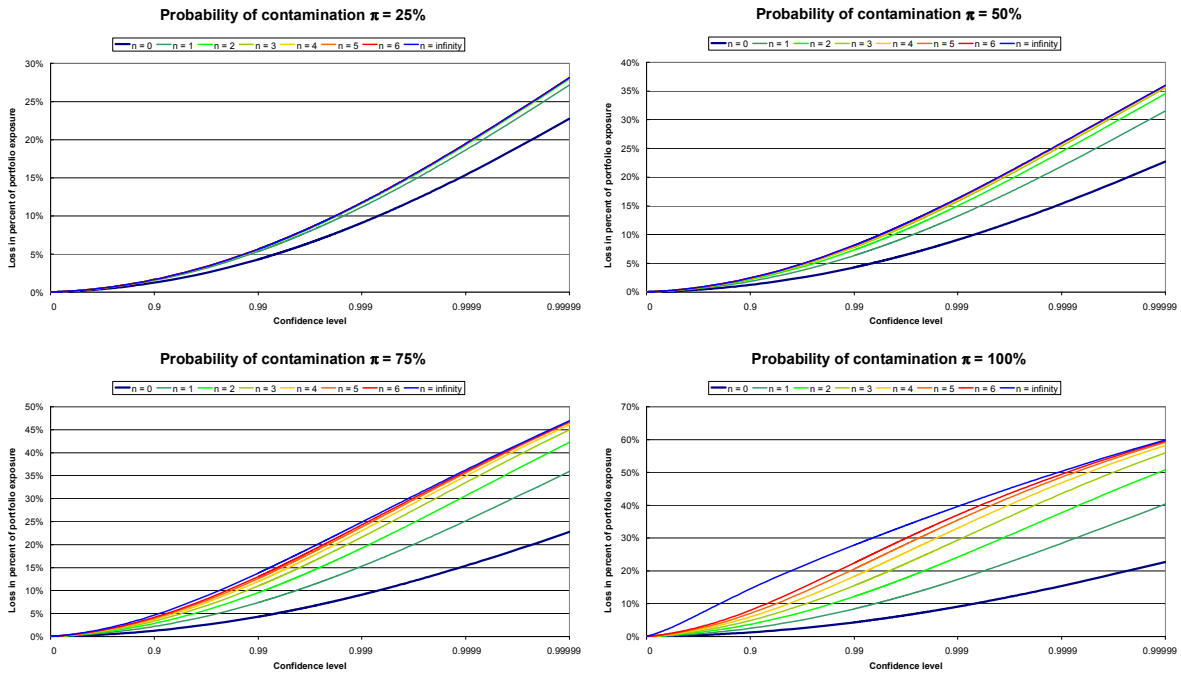

Figure 58: Cascading effect of microeconomic dependencies

Figure 58 gives an impression of the cascading effect of microeconomic dependencies on a homogenous portfolio[262]. For the first few rounds, the effect is significant even for small probabilities of contamination increasing drastically with the contamination probability. However, note that independent of the contamination probability the total cascading effect converges towards a limiting distribution if the number *n* of rounds tends to infinity.

262 Clients are assumed to have a default probability of 0.5% and risk index correlations of 20% in the normal correlation model. The portfolio was constructed so that each client affects exactly one other client, e.g. client 1 affects client 2, client 2 affects client 3 and so on. We define the contamination probability as the probability that a defaulting clients draws another clients who is related to him into insolvency as well.

Then, given a proportion of a_0 of defaulting clients in the portfolio, round *n* leads to an additional fraction of

$$a_{n+1} = \pi \cdot a_n \cdot \left(1 - \sum_{i=0}^{n} a_i\right)$$

of defaulting clients. The total cascading effect c_{n+1} is then

$$c_{n+1} = \sum_{i=0}^{n+1} a_i \ .$$

B. Time horizons for risk calculations

Bank portfolios usually comprise of credits and trades of strongly heterogeneous maturities. It is, therefore, not a matter of course to choose a single time horizon for portfolio risk calculations. Depending on the characteristics of the portfolio under consideration and the purpose of the risk analysis, a standardized time horizon for the entire portfolio or individual maturities for the single exposures and positions may be appropriate. Estimation of economic capital requirements puts the focus on a continuously renewing portfolio and, therefore, selects a normalized time frame for the entire portfolio. Conversely, e.g. the valuation of basket credit derivatives fixes a portfolio consisting of specific exposures and asks for potential losses until the maturity of the last position in the portfolio. In this context, a heterogeneous time scheme that is adapted to the maturity of the single exposures in the basket is suitable.

1. A standardized time horizon

If a portfolio analysis is performed in order to compute economic capital requirements an artificial, standardized time horizon for all portfolio components independent of their actual remaining time to maturity applies for two reasons:

Firstly, the fundamental concept of portfolio risk, all widely used risk measures such as expected loss, value at risk, shortfall and standard deviation are based upon, tracks the variation of portfolio losses within a certain amount of time. Hence, a standardized time frame is immanent to this notion of risk.

Secondly, portfolio structures are much more stable than isolated portfolio components. For instance, a six months money market loan ceases to exist after half a year. The total quantity of six months credits will habitually hardly have changed, though, because firms and financial institutions have a permanent need of short term finance. Figuratively speaking, the portfolio of a constantly operating bank can be seen as a lake that permanently changes its water, but remains more or less steady over time.

The choice of a standardized time horizon implicitly assumes that the portfolio manager is not predominantly interested in the single client's credit risk, even if it is possible to break down total portfolio risk to the level of a single client's risk contribution[263], but mainly in the risk level of the portfolio as a whole. For that reason, the practical supposition is made that a client who leaves the portfolio because his trades and credits have expired or because his position

[263] See below section II.D.1.

has been sold is always replaced by another client of approximately the same type, i.e. of the same credit worthiness, exposure, country, sector etc.

Following the accounting period and the rating update scheme, most banks choose a one-year time frame for their portfolio risk calculations. However, if the appropriate input data is used, e.g. default and transition probabilities, exposures, other intervals are possible. Figure 59 puts on view how the 99.9%-value at risk increases with the time horizon for various one-year probabilities of default.

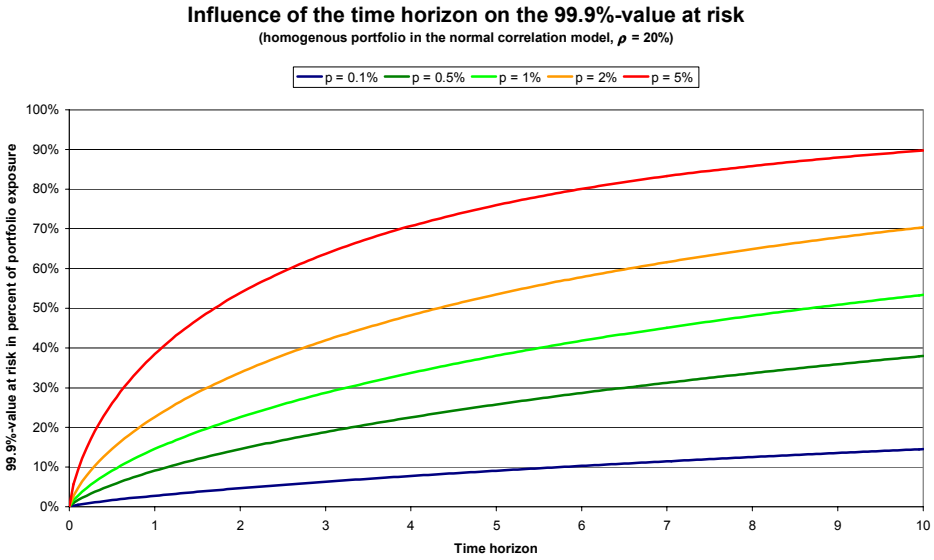

Figure 59: Influence of the time horizon on the 99.9%-value at risk

In order to be able to consider long time horizons, it is essential to be capable of providing the corresponding default probabilities. In the exhibit, we chose one-year default probabilities p and extrapolated them to arbitrary time horizons assuming that clients' defaults follow a Markov process, i.e. that the probability to go into bankruptcy in the time interval from t to t+1 is constant at $p = p_1$ provided that a client did not default until time t. A client's probability p_t to default in the time interval between 0 and t then is

$$p_t = 1 - \left(1 - p_1\right)^t .$$

In reality, this Markov assumption, that greatly facilitates the derivation of long term default probabilities, is only a very rough approximation of clients' true default processes as a comparison of the resulting probabilities with directly estimated probabilities, e.g. by rating agencies, reveals.

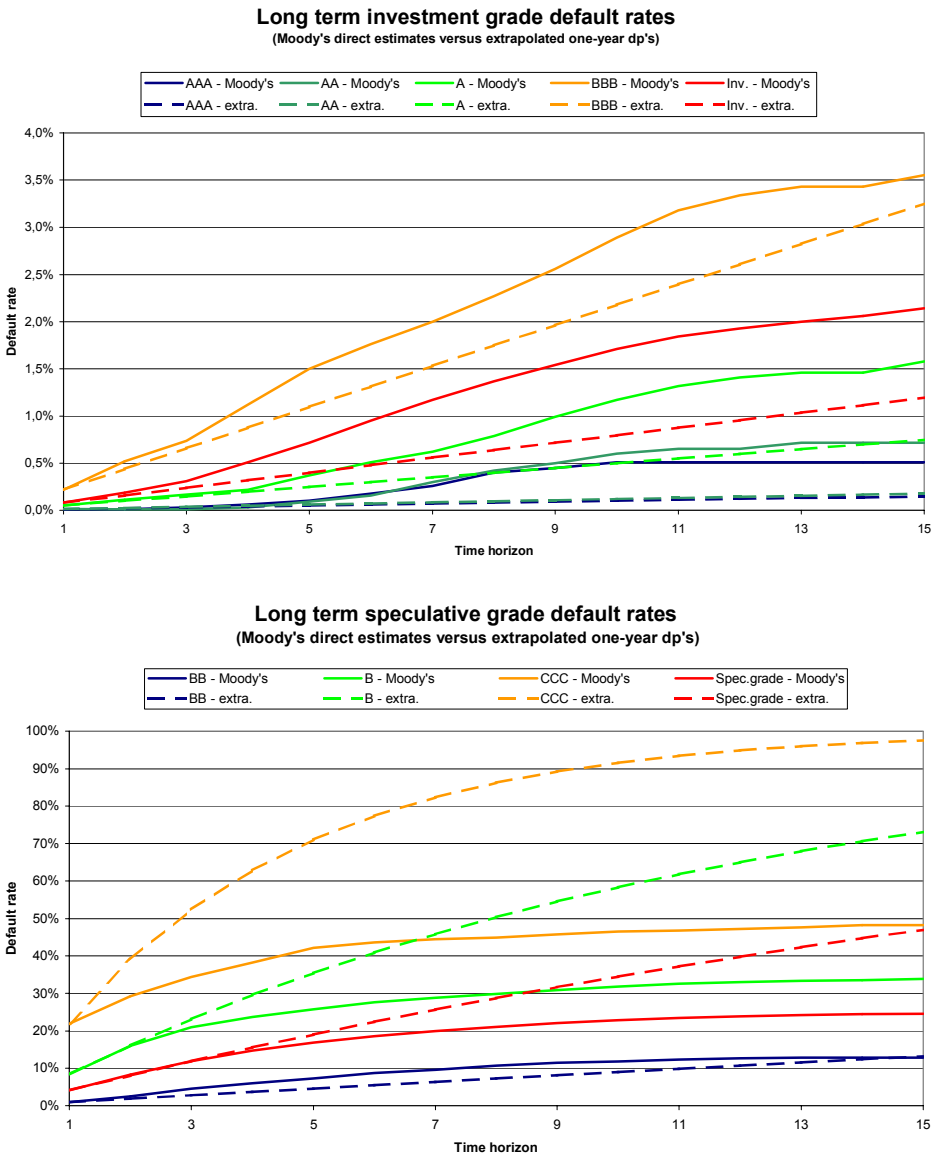

Figure 60: Term structure of default rates: direct estimates versus extrapolated values

Figure 60 compares Moody's estimates[264] of long term default rates with the respective extrapolated values. The extrapolated default probabilities turn out to be consistently and considerably lower than the directly estimated values for investment grade ratings up to BB while they are significantly higher for speculative grade firms. Note that this deviation inevitably carries over to the calculation of portfolio risk.

This example shows that long time horizons should only be chosen very carefully because the direct estimation of long term default rates, which is also essential to calibrate extrapolation techniques, requires an increasingly long history of default experience. Otherwise the precision of portfolio risk calculations would suffer to a great extent.

[264] See Moody's 2000, p. 16, table 11.

2. Heterogeneous time horizons

A very different situation is given if the portfolio does not permanently renew itself, but is a fixed set of gradually expiring exposures. This is, for instance, the case of basket credit derivatives. Here, a financial institution sells the losses coming off a certain portfolio of exposures, usually except a 'first loss piece' that still has to be carried by the vendor of the derivative.

A typical way to value this kind of credit derivative is to calculate the basket's expected excess loss over the first loss piece until maturity of the last position in the basket, i.e. the expected shortfall corresponding to the first loss piece. Here a standardized time horizon would apparently distort the results. Instead, risk calculations have to be performed successively over multiple periods until the last maturity has been reached which includes in each step only those exposures that don't have expired or defaulted before. Note, however, that the difficulties in estimating long term default probabilities, that were described in the previous section, naturally remain the same in this context.

C. Quantification of portfolio risk

Having specified the clients' default probabilities, exposures, recovery rates and interdependencies and the relevant time horizon for each position and the entire portfolio, the portfolio model is fully defined. The next task is the quantification and analysis of portfolio risk through the portfolio loss distribution and various risk measures, which focus on certain characteristics of the loss distribution.

We begin the section with a discussion of the calculation of the portfolio loss distribution and the construction of confidence intervals. We will then define risk measures based upon the previously derived loss distribution, show how they can be estimated and used in practice and discuss the properties of the estimators.

1. Portfolio loss distribution

The main result of the quantitative portfolio risk analysis is the calculation of the portfolio loss distribution or, more precisely, of the cumulative portfolio loss distribution function, i.e. of the function

$$\mathrm{L}(x) = \mathbf{P}\{\text{portfolio loss} \le x\}.$$

The loss distribution contains all information about portfolio risk at the aggregate level and is also a central component of the analysis of the composition of portfolio risk.

In most portfolio models, the analytical calculation of the loss distribution is tremendously difficult due to the complex interrelations of portfolio components. However, the loss distribution can be approximately derived through Monte-Carlo-simulation techniques. Here, *n* random portfolio losses $X_1,...,X_n$ are drawn as independent realizations from the portfolio loss distribution and the empirical loss distribution function is calculated as

$$\mathrm{L}_n(x)=\frac{1}{n}\sum_{i:X_i\le x}1.$$

Note that $\mathrm{L}_n(x)$ depends on the simulated random portfolio losses $X_1,...,X_n$ and is, therefore, random itself. Nonetheless, the empirical loss distribution function can be used to estimate the portfolio loss distribution because it uniformly converges towards the true portfolio loss distribution with probability one:

Theorem 13 (Glivenko-Cantelli):
Let $X_1,...,X_n$ be independent realizations of the random variable X with cumulative distribution function $\mathrm{L}(x)$. Let $\mathrm{L}_n(x)$ be a realization of the empirical distribution function implied by $X_1,...,X_n$. Then

$$\mathbf{P}\left\{\lim_{n\to\infty}\sup_{-\infty<x<+\infty}|\mathrm{L}_n(x)-\mathrm{L}(x)|=0\right\}=1\ .$$

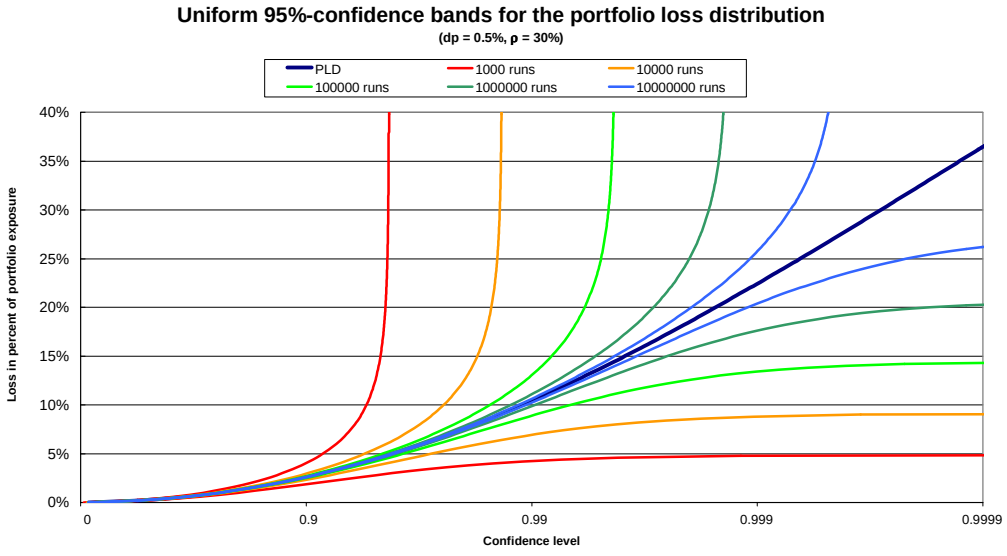

Figure 61: Uniform 95%-confidence bands for the portfolio loss distribution dependent on the number of simulation runs

Figure 61 shows the portfolio loss distribution of a homogenous portfolio in the normal correlation model[265] and uniform confidence bands for the empirical loss distribution dependent on the number of simulation runs[266]. The confidence bands can be interpreted that with a probability of 95% the *entire* empirical loss distribution will be within the region between the upper and the lower band. It is evident from the exhibit that the probability of small portfolio losses can well be estimated even with a small number of draws from the loss distribution. Conversely, in order to get a good fit at high percentiles of the loss distribution, a comparatively large number of simulation runs is necessary.

We get a qualitatively similar, but quantitatively more moderate result, if we do not look at uniform confidence bands, i.e. at the supreme of the deviations of the empirical from the true loss distribution over all x, but at the pointwise deviation of $\mathrm{L}_n(x)$ from $\mathrm{L}(x)$.

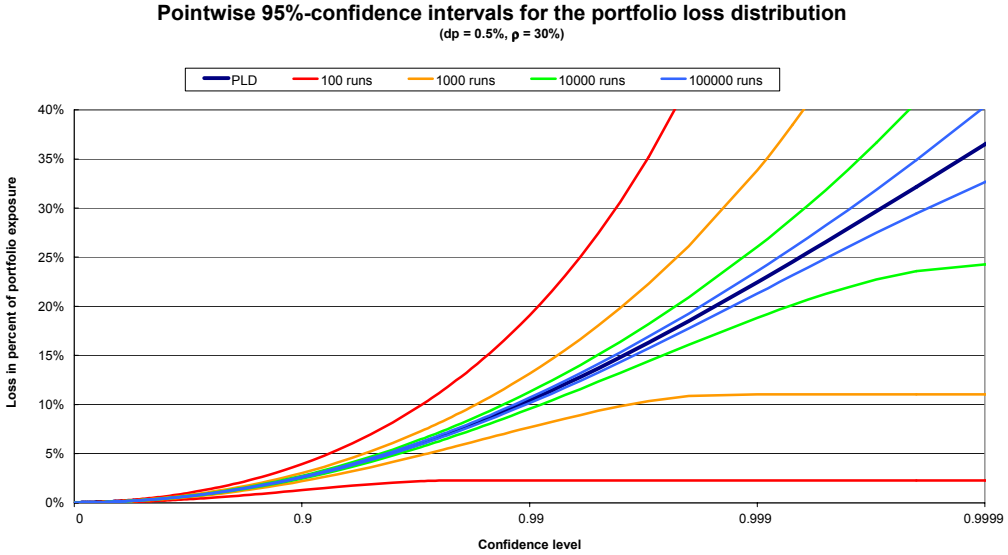

Figure 62: Pointwise 95%-confidence intervals for the portfolio loss distribution dependent on the number of simulation runs

Here, the focus is not at the entire loss distribution at once, but rather at every percentile of the distribution separately. The confidence intervals state that a given percentile of the empirical loss distribution will lie in the specified range with a probability of 95%. It may happen, though, that an empirical distribution function is inside the confidence intervals for some x while it is outside for others.

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265 We assume that the portfolio exposure is split among an infinite number of clients. See above Theorem 2.

266 The asymptotic portfolio loss distribution is absolutely continuous with respect to Lebesgue-measure so that the statistic

$$d_n = \sup_{x \in \mathbf{R}} | L_n(x) - L(x) |$$

has a Kolmogorov-Smirnov-distribution, which can be used to calculate the confidence bands.

We see that the pointwise convergence of the empirical loss distribution to the true one occurs much faster than the uniform convergence especially in the tail of the distribution. The exact speed of convergence for the case of continuous loss distributions is stated in the next theorem.

We begin with a definition.

Definition:
Let

$$X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$$

be the order statistic of the random draws from the loss distribution, and let

$$\mathrm{L}_n^-(p) := \begin{cases} X_{(pn)} & \text{if } pn \in \mathbf{N} \\ X_{(\lfloor pn \rfloor + 1)} & \text{else} \end{cases}$$

be the empirical inverse cumulative portfolio loss distribution function[267] for $0 < p < 1$.

Theorem 14 (Bassi, Embrechts, Kafetzaki) [268]*:*
Assume that the true portfolio loss distribution L is continuous with density $f(x)$ and inverse cumulative distribution function $\mathrm{L}^{-1}(p)$ for $0 < p < 1$. Then

$$\mathrm{L}_n^-(p) \sim \mathsf{N}\left(\mathrm{L}^{-1}(p); \frac{p(1-p)}{n \cdot f^2\left(\mathrm{L}^{-1}(p)\right)} \right)$$

for large n if $f\left(\mathrm{L}^{-1}(p)\right) > 0$.

Thus, the pointwise estimation of the portfolio loss distribution is not only consistent in n as was already implied by Theorem 13, but the stochastic convergence is of order $O\left(\frac{1}{\sqrt{n}}\right)$.

However, the variance of the percentile estimator $\mathrm{L}_n^-(p)$ does not only crucially depend on the number of simulation runs n, but also on the value of the density of the loss distribution at the true percentile $\mathrm{L}^{-1}(p)$. Since the likelihood $1-p$ of extremely large portfolio losses is typically very small in credit risk management with equally small values of $f\left(\mathrm{L}^{-1}(p)\right)$, this implies that the efficiency of the pointwise estimation of the portfolio loss distribution usually

[267] $\lfloor np \rfloor$ is defined as the largest integer $m \le np$.

[268] See Bassi et al. 1998, p. 117.

declines in p for values of p close to 1 leading to larger confidence intervals for a given number of simulation runs (see Figure 62).

In the above examples, we assumed that the portfolio loss distribution is known and continuous and, in order to illustrate the natural variation in simulation results, we derived analytic confidence bands and confidence intervals that comprise the simulated loss distribution with a given probability.

In practical applications, we have the opposite situation that the true loss distribution is unknown and usually discontinuous. In this case, pointwise (random) confidence intervals for the true loss distribution can be calculated from the simulated loss distribution.

For any $p \in (0, 1)$ the p-percentile of a general loss distribution can be estimated through the inverse empirical cumulative loss distribution defined above. Let the random variable $Y = 0$ if a random draw from the loss distribution is smaller than or equal to $\mathrm{L}^{-1}(p)$. By definition[269] of the p-percentile,

$$\mathbf{P}\{Y = 0\} = q \geq p\,.$$

If the loss distribution is continuous, $q = p$ for all p. If, on the other hand, the loss distribution has discrete parts, there may be some p so that $q > p$. These are the portfolio losses x where the loss distribution L has point masses, namely the discontinuity points of L. Since we do not know the true loss distribution, the exact value of q is unknown in these cases.

Nevertheless, we can conclude that the number of random draws from the loss distribution that are smaller than or equal to $\mathrm{L}^{-1}(p)$ is binomially distributed with parameters q and n, if the experiment is repeated n times. Hence, a confidence interval for $\mathrm{L}^{-1}(p)$ with confidence level α can be constructed by choosing indices j and k so that

$$\mathbf{P}\{X_{(j)} \leq L^{-1}(p) < X_{(k)}\} = \sum_{i=j}^{k-1} \binom{n}{k} q^i (1-q)^{n-i} \geq \alpha$$

with $X_{(j)} \leq L_n^-(p) < X_{(k)}$. These confidence intervals would be exact if we knew the value of q. We can use these exact confidence intervals that use the unknown parameter q to derive approximate and slightly conservative confidence intervals that are dependent on p and α alone.

[269] The p-percentile is defined as $\mathrm{L}^{-1}(p) := \inf\{x : \mathbf{P}\{\text{loss} \leq x\} \geq p\}$.

For large[270] n the binomial distribution can be approximated by the normal distribution since

$$\lim_{n\to\infty}\mathbf{P}\left\{\frac{\sum_{i=1}^{n}Y_i - nq}{\sqrt{nq(1-q)}} \le x\right\} = \Phi(x)$$

due to the central limit theorem. With

$$a = 1-(1-\alpha)/2$$
$$z_a = \Phi^{-1}(a)$$

and functions

$$c(p,n) := \begin{cases} pn & \text{if } pn \in \mathbf{N} \\ \lfloor pn \rfloor + 1 & \text{else} \end{cases}$$

$$d(p,n) := \begin{cases} c(p) - z_a \cdot \sqrt{np(1-p)} & \text{if } z_a \cdot \sqrt{np(1-p)} \in \mathbf{N} \\ \lfloor c(p) - z_a \cdot \sqrt{np(1-p)} \rfloor & \text{else} \end{cases}$$

$$e(p,n) := \begin{cases} c(p) + z_a \cdot \sqrt{np(1-p)} & \text{if } z_a \cdot \sqrt{np(1-p)} \in \mathbf{N} \\ \lfloor c(p) + z_a \cdot \sqrt{np(1-p)} \rfloor + 1 & \text{else} \end{cases}$$

symmetric confidence intervals for $\mathrm{L}^{-1}(p)$ with confidence level α are implied from

$$\mathbf{P}\left\{X_{(d(q,n))} \le L^{-1}(p) < X_{(e(q,n))}\right\} \ge \alpha .$$

Note that $\mathrm{L}_n^-(p) = c(p,n)$ is a consistent estimator of $\mathrm{L}^{-1}(p)$ due to Theorem 13. Moreover, the standard deviation $\sqrt{np(1-p)}$ of the asymptotic normal distribution of $\sum_{i=1}^{n} Y_i$ falls strictly monotonously in $p \ge 1/2$ so that

$$\sqrt{np(1-p)} \ge \sqrt{nq(1-q)}$$

for $p \ge 1/2$. Conservative symmetric confidence intervals for $\mathrm{L}^{-1}(p)$ with confidence level α are, hence, given as

$$\mathbf{P}\left\{X_{(d(p,n))} \le L^{-1}(p) < X_{(e(p,n))}\right\} \ge \alpha .$$

[270] As a rule of thumb, the approximation works well if $np(1-p) > 9$.

For $p < 1/2$ the calculation of confidence intervals is less problematic because this part of the portfolio loss distribution is much less relevant for risk management purposes. In addition, loss distributions are usually more continuous in the region of low losses than in the tail. This means that the above approximation is unlikely to largely underestimate confidence intervals when used for values of $p < 1/2$. With

$$g(p,n) := \begin{cases} c(p) - z_a \cdot \sqrt{n}/2 & \text{if } z_a \cdot \sqrt{n}/2 \in \mathbf{N} \\ \lfloor c(p) - z_a \cdot \sqrt{n}/2 \rfloor + 1 & \text{else} \end{cases}$$

$$h(p,n) := \begin{cases} c(p) + z_a \cdot \sqrt{n}/2 & \text{if } z_a \cdot \sqrt{n}/2 \in \mathbf{N} \\ \lfloor c(p) + z_a \cdot \sqrt{n}/2 \rfloor + 1 & \text{else} \end{cases}$$

conservative confidence intervals can be obtained for $p < 1/2$ as

$$\mathbf{P}\left\{X_{(g(p,n))} \le L^{-1}(p) < X_{(h(p,n))}\right\} \ge \alpha .$$

In order to condense and to interpret the information contained in the portfolio loss distribution, risk measures are defined that are entirely based upon the previously calculated loss distribution and are conceptually identical in all risk models. In the following sections we define and discuss the expected loss, the portfolio standard deviation, the value at risk, and the shortfall.

2. Expected loss

The expected portfolio loss is the most elementary risk measure. It is defined as the expected value of the portfolio loss distribution. The most important characteristic of the expected loss is that it is always the sum of the expected losses of the portfolio components and, thus, cannot be diversified. This is why it can be considered as the basic risk of a portfolio that has to be covered by risk premiums under all circumstances, if the financial institution does not want to expect a loss from its portfolio that is not accessible to risk management otherwise.

a) Estimation

Let $X_1, \ldots, X_n$ be the simulated independent draws from the portfolio loss distribution. Then the expected loss μ can be unbiasedly and consistently estimated through the arithmetic mean of the simulated loss distribution:

$$\hat{\mu}(X_1,..., X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

and we have

$$\text{(unbiasedness)} \qquad \mathbf{E}[\hat{\mu}(X_1,..., X_n)] = \frac{1}{n}\sum_{i=1}^{n}\mathbf{E}[X_i] = \mu$$

and

$$\text{(consistency)} \qquad \lim_{n\to\infty}\mathbf{P}\{|\hat{\mu}(X_1,..., X_n) - \mu| > \varepsilon\} = 0$$

for all $\varepsilon > 0$.

b) Confidence intervals

It follows from the central limit theorem, that $\hat{\mu}(X_1,..., X_n)$ is asymptotically normally distributed with mean μ and standard deviation

$$\hat{\sigma}_{\hat{\mu}} = \frac{1}{n}\sqrt{\sum_{i=1}^{n}\sigma^2(X_i)} = \frac{1}{n}\sqrt{n\sigma_X^2} = \frac{\sigma_X}{\sqrt{n}}$$

where σ_X is the standard deviation of the portfolio loss distribution. σ_X can be consistently estimated as

$$\hat{\sigma}_X = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}(X_1,..., X_n))^2}$$

(see section II.C.3.a) below).

Upper and lower limits of confidence intervals at the confidence level α for the expected loss then are

$$\mathrm{CI}_\mu = \hat{\mu}(X_1,..., X_n) \pm z_a \cdot \hat{\sigma}_{\hat{\mu}}$$

with

$$a = 1 - (1-\alpha)/2 \text{ and } z_a = \Phi^{-1}(a).$$

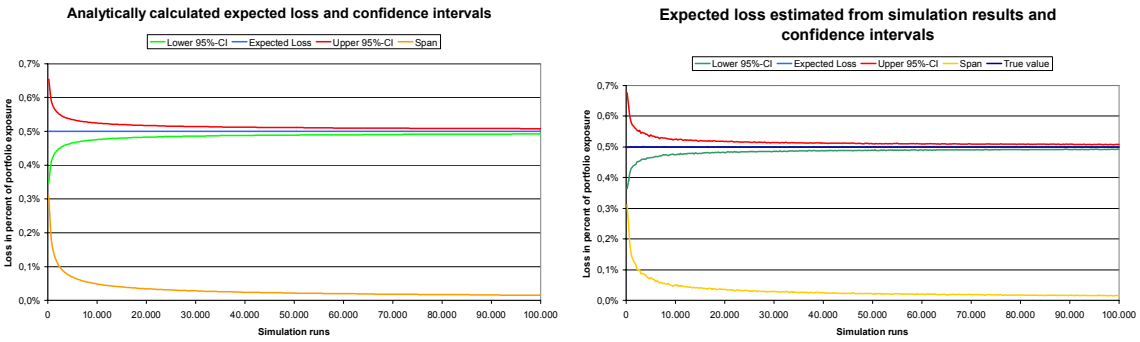


Figure 63: Accuracy of the estimation of the expected portfolio loss dependent on the number of simulation runs

Figure 63 shows the analytic and simulated results for the expected loss and confidence bands for the estimator at the 95%-confidence level for a homogenous portfolio with default probability 0.5% and risk index correlations of 30% in the normal correlation model. The span is defined as the difference between the upper and the lower bound of the confidence interval. To get the simulated values of the expected loss and its confidence intervals in the right chart, the distribution of the estimator was simulated 1,000 times and the mean, the 2.5%-percentile and the 97.5%-percentile were plotted[271].

It is evident from the exhibits that the simulation results are well in line with the analytic solution and that the estimation of the expected loss is unbiased for all numbers of simulation runs. Note as well that the span of the confidence intervals decreases strictly monotonously in the number of simulation runs as would be expected from the consistency of the estimator, but that the increase in precision from an additional run declines rapidly.

3. Standard deviation[272]

The portfolio standard deviation is another widely used risk measure. It is defined as the standard deviation of the portfolio loss distribution. Other than the expected loss, the size of the portfolio standard deviation changes with the composition of the portfolio even if the portfolio's expected loss is held constant. Hence, by considering clients' marginal contributions to the portfolio standard deviation, the standard deviation can be used to analyze the portfolio structures and to localize components that are well diversified in the portfolio and others where risk is particularly concentrated.

271 Two simulations were performed sequentially. First, a sample of *n* points was drawn from the portfolio loss distribution and the expected value was estimated as described above. Then, this was repeated 1,000 times in order to obtain 1,000 independent estimates for the expected loss as an empirical proxy for the distribution of the estimator.

272 On the applicability of the portfolio standard deviation in credit risk management confer to Wehrspohn 2001.

One of the major advantages of the standard deviation that founded its popularity among practitioners is that it is analytically computable in elementary portfolio models such as the Vasicek-Kealhofer model[273]. In more complicated models such as the CRE model where several influences on the clients' credit risk superimpose one another such as country risk, sectorial correlations and individual dependencies with other clients, the analytical calculation of the portfolio standard deviation is considerably more complicated and no longer practicable. Below we show how the standard deviation and confidence intervals can be estimated based on the simulation results.

A second great advantage of the portfolio standard deviation is that it expresses a feature of the entire loss distribution. This is particularly valuable if it comes to the calculation of the individual clients' contributions to total portfolio risk because for a client with positive probability of default and positive exposure his marginal standard deviation is also positive. This is not necessarily the case with a client's marginal value at risk or shortfall which much stronger depend on the specific simulation results and the number of simulation runs performed since they only consider a specific percentile or the tail of the loss distribution, respectively.

However, in all credit risk models, the standard deviation has the important drawback that it cannot be interpreted easily. This is a decisive difference between the application of portfolio standard deviation in the risk management of, say, equity portfolios as compared to the risk management of credit portfolios. In popular equity portfolio models such as the capital asset pricing model (CAPM), the distribution of portfolio returns is *always* a normal distribution so that there is a constant ratio between portfolio standard deviation and any value at risk to a constant confidence level. For example, no matter how the portfolio is composed, the 99%-value at risk is always 2.32 standard deviations above the mean in a normally distributed set-

[273] In general, the portfolio variance is the sum of the variances and covariances of the portfolio components and can be written as

$$\sigma^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\sigma_{ij}^2 = \sum_{i=1}^{k}\sum_{j=1}^{k}\rho_{ij}^{def}\sqrt{\eta_i\eta_j p_i(1-p_i)p_j(1-p_j)}$$

where *k* is the number of clients in the portfolio, p_i is the default probability of client *i* for $i = 1,...,k$, η_i his exposure, and ρ_{ij}^{def} the default correlation between client *i* and *j*. In the Vasicek-Kealhofer model the default correlation ρ_{ij}^{def} can be directly computed as

$$\rho_{ij}^{def} = \frac{\Phi_2\left(\Phi^{-1}(p_i), \Phi^{-1}(p_j), \rho_{ij}\right) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}}.$$

Here $\Phi_2(\cdot,\cdot,\rho)$ is the bivariate normal cumulative distribution function and ρ_{ij} the risk index correlation between client *i* and *j*.

ting. The portfolio standard deviation and portfolio value at risk are, therefore, exchangeable in their informational content so that the standard deviation inherits the interpretability of the value at risk.

In credit portfolio management things are different. Here the portfolio loss distribution does not have a fixed shape, but permanently substantially changes with the composition of the portfolio and the quantitative and qualitative structure of the risk factors. As a consequence, the relationship between standard deviation and portfolio value at risk is not predetermined. It is not even monotonous, meaning the value at risk may decrease while the standard deviation increases.

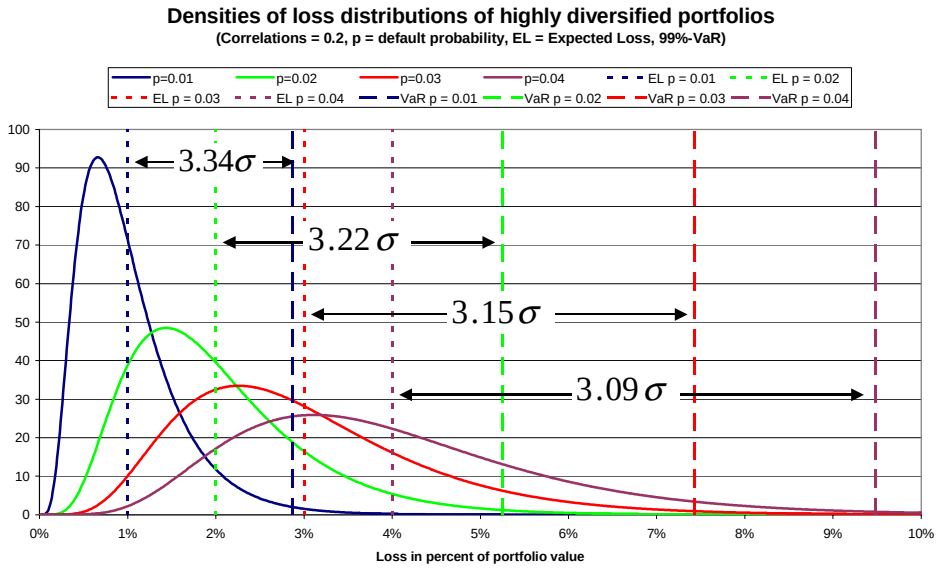


Figure 64: The ratio of value at risk and standard deviation of credit portfolios

Figure 64 shows the ratio of the 99%-value at risk to the portfolio standard deviation in homogenous portfolios in the normal correlation model. In this example, the ratio decreases while the clients' default probability increases. Hence, even in the highly idealized case of the normal correlation model, the standard deviation cannot be used as a proxy for the value at risk and is a rather blurred measure for the variability and the width of a distribution than a clear-cut measure of portfolio risk.

a) Estimation

Let $X_1,...,X_n$ be the simulated independent draws from the portfolio loss distribution. Then the portfolio variance σ^2 can be unbiasedly and consistently estimated through the variance of the simulated loss distribution. Let

$$\hat{\mu} = \hat{\mu}(X_1,..., X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

be the mean of the simulated loss distribution. Then

$$\hat{\sigma}^2(X_1,..., X_n) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$

and we have

$$\begin{aligned}
\mathbf{E}\left[\hat{\sigma}^2(X_1,..., X_n)\right] &= \frac{1}{n-1}\left[\left(\sum_{i=1}^{n}\mathbf{E}\left[X_i^2\right]\right) - n\mathbf{E}\left[\hat{\mu}^2\right]\right] \\
\text{(unbiasedness)} \qquad &= \frac{1}{n-1}\left[\left(\sum_{i=1}^{n}\left(\sigma^2+\mu^2\right)\right) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right] \\
&= \frac{1}{n-1}\left(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right) = \sigma^2
\end{aligned}$$

It is mere algebra to show that the variance of the estimator $\hat{\sigma}^2$ is given as

$$\begin{aligned}
\mathbf{Var}\left(\hat{\sigma}^2(X_1,..., X_n)\right) &= \mathbf{E}\left(\hat{\sigma}^4\right) - \sigma^4 \\
&= \frac{(n-2)\kappa}{(n-1)^2} + \frac{\kappa}{n(n-1)^2} - \frac{(n-3)\sigma^4}{n(n-1)}
\end{aligned}$$

where κ is the kurtosis of the portfolio loss distribution defined as

$$\kappa = \mathbf{E}(X-\mu)^4 .$$

Then

$$\lim_{n\to\infty}\mathbf{Var}\left(\hat{\sigma}^2(X_1,..., X_n)\right) = 0$$

implying

$$\text{(consistency)} \qquad \lim_{n\to\infty}\mathbf{P}\left\{\left|\hat{\sigma}^2(X_1,..., X_n) - \sigma^2\right| > \varepsilon\right\} = 0$$

for all $\varepsilon > 0$. Due to the concavity of the square-root-transformation, the portfolio standard deviation can be consistently estimated with a small sample bias $\mathrm{E}\left(\hat{\sigma}(X_1,..., X_n)\right) < \sigma$.

b) Confidence intervals

It follows from the central limit theorem, that $\hat{\sigma}^2(X_1,..., X_n)$ is asymptotically normally distributed with mean σ^2. The variance of the estimator was already calculated in the previous section. Note that for large *n* the variance approximately simplifies to

$$\mathbf{Var}\left(\hat{\sigma}^2\left(X_1,\ldots,X_n\right)\right)\approx\frac{\kappa-\sigma^4}{n}$$

and that the kurtosis κ of the portfolio loss distribution can be consistently estimated as

$$\hat{\kappa}\left(X_1,\ldots,X_n\right)=\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\hat{\mu}\right)^4$$

from the simulation results.

Upper and lower limits of two-sided confidence intervals for the portfolio standard deviation at the confidence level α then are

$$\mathrm{CI}_\sigma=\sqrt{\hat{\sigma}^2\pm z_a\cdot\sqrt{\frac{\hat{\kappa}-\left(\hat{\sigma}^2\right)^2}{n}}}$$

with

$$a=1-\left(1-\alpha\right)/2\ \text{ and }\ z_a=\Phi^{-1}\left(a\right).$$

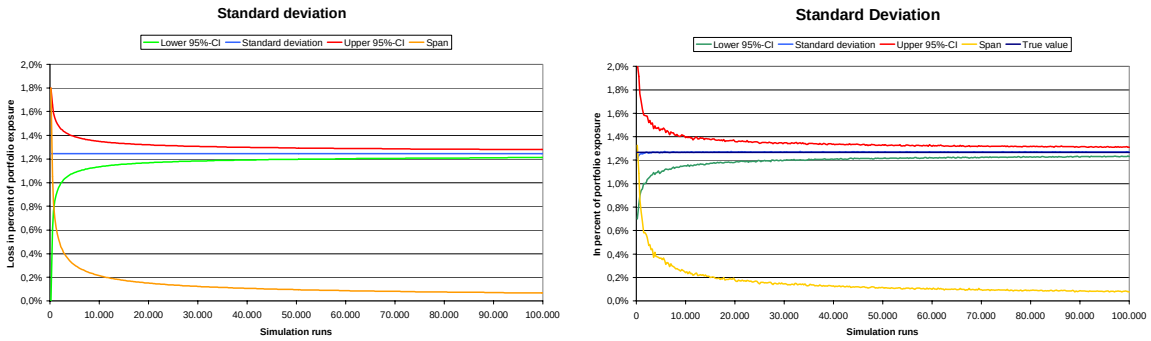


Figure 65: Accuracy of the estimation of the standard deviation of portfolio losses dependent on the number of simulation runs

Figure 65 shows the analytic and simulated results for the portfolio standard deviation and confidence bands for the estimator at the 95%-confidence level for a homogenous portfolio with default probability 0.5% and risk index correlations of 30% in the normal correlation model. The span is defined as the difference between the upper and the lower bound of the confidence interval. To get the simulated values of the portfolio standard deviation and its confidence intervals in the right chart, the distribution of the estimator was simulated 1,000 times and the mean, the 2.5%-percentile and the 97.5%-percentile were plotted[274].

[274] Two simulations were performed sequentially. First, a sample of *n* points was drawn from the portfolio loss distribution and the standard deviation was estimated as described above. Then, this was repeated 1,000 times in order to obtain 1,000 independent estimates for the standard deviation as an empirical proxy for the distribution of the estimator.

Similar to the estimation of expected portfolio losses, it is evident from the exhibits that the simulation results are well in line with the analytic solution and that the estimation of the portfolio standard deviation is consistent. Note that the small sample bias of the estimation already disappears after a few thousand simulation runs. The span of the confidence intervals decreases strictly monotonously in the number of simulation runs as should be expected from the consistency of the estimator, though the increase in precision from an additional run declines rapidly.

4. Value at risk

The portfolio value at risk is the most important risk measure in modern credit risk management. It is defined at a percentile of the portfolio loss distribution at a certain confidence level of, say, 99% or 99.9%. It is popular for two main reasons. Firstly, the value at risk can be interpreted as the smallest portfolio loss that will not be exceeded with a given probability. Through the corresponding confidence level, it can, thus, be used to define a certain risk policy that may, for instance, be reflected in the amount of economic capital a financial institution supplies to cover its portfolio risk. In this application, the value at risk states the most economical way to allocate bank-owned capital and still reach a predefined security level.

Secondly, as the portfolio standard deviation, the value at risk reflects the portfolio structure and changes with its composition also for a given expected loss. For this reason, the value at risk can be used to calculate risk and risk contributions at all portfolio levels consistently.

From a theoretical point of view, the value at risk has recently been criticized because it is no coherent risk measure in the sense of Artzner et al.[275] since it is not sub-additive. For example, we may have for two portfolios P_1 and P_2

$$VaR(P_1 \cup P_2) > VaR(P_1) + VaR(P_2).$$

As an example, assume two exposures E_1 and E_2 with identical loss distributions

$$\begin{aligned} \mathbf{P}\{\text{loss}(E_i) = 0\} &= 98\% \\ \mathbf{P}\{\text{loss}(E_i) = 1\} &= 1\% \\ \mathbf{P}\{\text{loss}(E_i) = 100\} &= 1\% \end{aligned}$$

for $i = 1,2$. Suppose further that losses from both exposures are dependent in the following way

[275] Artzner et al., 1998, definition 2.4, p.7.

$$\mathbf{P}\{\text{loss}(E_1)=0 \text{ AND } \text{loss}(E_2)=0\}=98\%$$
$$\mathbf{P}\{\text{loss}(E_1)=1 \text{ AND } \text{loss}(E_2)=100\}=1\%$$
$$\mathbf{P}\{\text{loss}(E_1)=100 \text{ AND } \text{loss}(E_2)=1\}=1\%$$

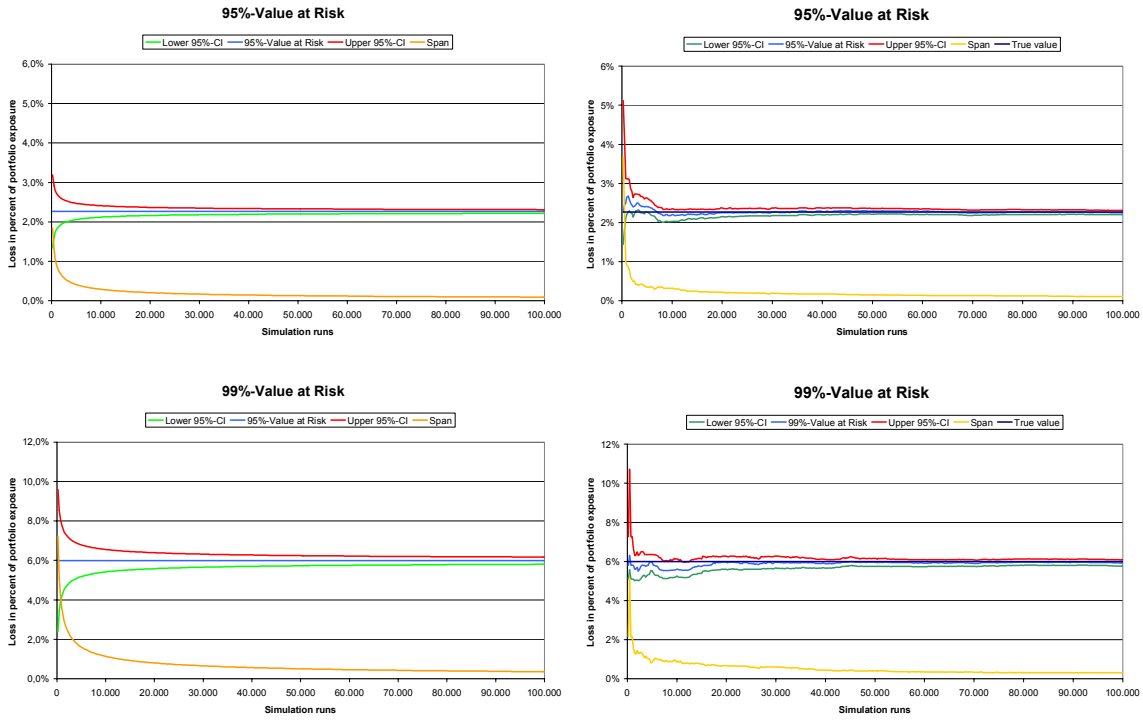
Then the 99%-values at risk of two portfolios P_1 and P_2 that only contain exposures E_1 or E_2, respectively, are

$$VaR(P_1)=VaR(P_2)=1.$$

The 99%-value at risk of the joint portfolio $P_1 \cup P_2$, however, is

$$VaR(P_1 \cup P_2)=101>2=VaR(P_1)+VaR(P_2).$$

The non-sub-additivity of the value at risk measure implies that a bank – theoretically – could manipulate its risk statement and reduce the reported risk by dividing the total bank portfolio into several smaller sub-portfolios and adding up the resulting component risks. In real world situations this procedure has no significance.



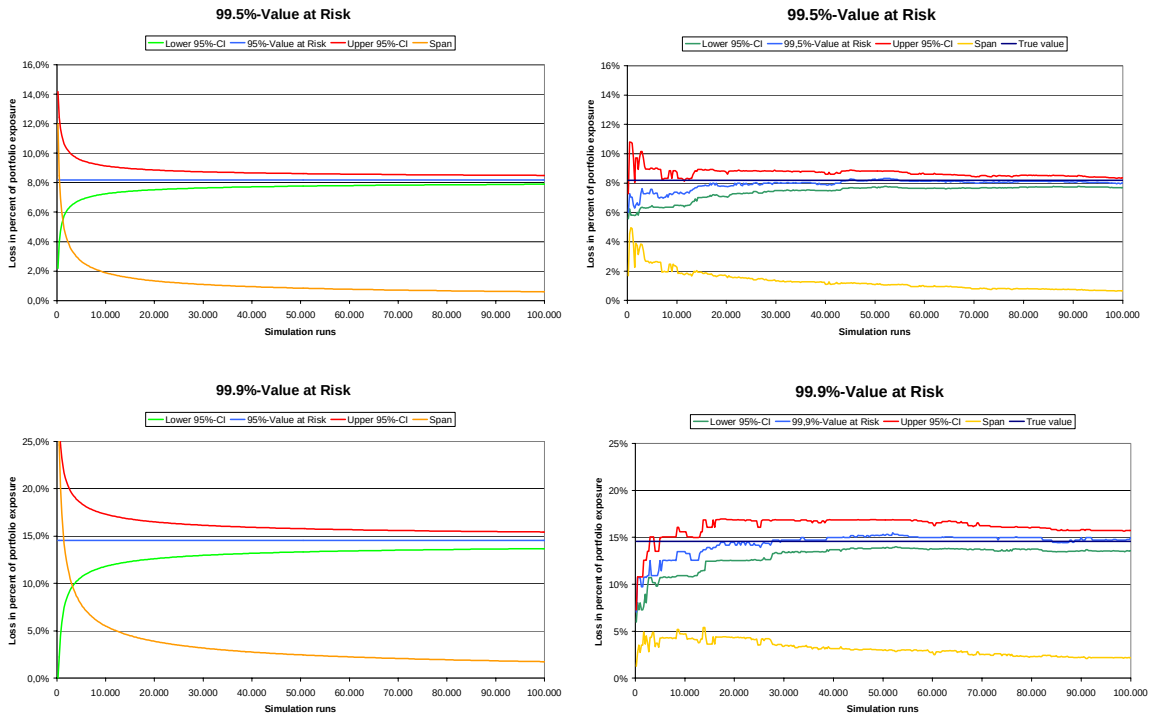


Figure 66: Accuracy of the estimation of the value at risk of portfolio losses dependent on the number of simulation runs

The methodology of the estimation of the value at risk and its confidence intervals was already stated in section II.C.1. above. Figure 66 gives an example of analytic and simulated results for the portfolio value at risk at various confidence levels and confidence bands for the estimator at the 95%-confidence level for a homogenous portfolio with default probability 0.5% and risk index correlations of 30% in the normal correlation model. To obtain the simulated values of the value at risk and its confidence intervals in the right charts, the distribution of the estimator was simulated 1,000 times and the mean, the 2.5%-percentile and the 97.5%-percentile were plotted [276].

The main difference to the estimation of expected loss is that the value at risk-estimator is consistent (as was also shown in section II.C.1), but not unbiased for high percentiles and low numbers of simulation runs. While the 95%- and 99%-value at risk may already be unbiasedly estimated if 10.000-20.000 simulation runs were performed, the 99.9%-value at risk requires at least 50.000 runs. This shows that the choice of the number of simulation runs is not only relevant for variance reduction in the estimation, but also for the elimination of estimation-biases.

[276] Two simulations were performed sequentially. First, a sample of *n* points was drawn from the portfolio loss distribution and the value at risk was estimated as described above. Then, this was repeated 1,000 times in order to obtain 1,000 independent estimates for the value at risk as an empirical proxy for the distribution of the estimator.

5. Shortfall

The portfolio shortfall is a risk measure that is derived from the value at risk. It is defined as the expected portfolio loss under the condition that the portfolio loss is larger than the value at risk.

The shortfall has gained popularity among researchers because it can be shown that it is coherent in the sense of Artzner et al.[277]. Particularly, other than the value at risk from which it is derived, it is sub-additive. For two portfolios P_1 and P_2 it is always

$$S(P_1 \cup P_2) \leq S(P_1) + S(P_2).$$

In practice, banks use the shortfall to assess the consequences of the case that is left open by the value at risk, i.e. of the large losses that occur only with 1%- or 0.1%-probability and may seriously weaken the financial structure of a bank.

A second important application of the shortfall is the valuation of basket credit derivatives. In a basket credit derivative a bank sells the credit risk of a portfolio of exposures, but usually keeps a so called 'first loss piece'. This means that the vendor of the credit derivative still has to cover losses up to a certain level himself, while the buyer of the credit derivative only pays for losses in excess of the first loss piece. The value of the credit derivative then is the expected shortfall of the basket portfolio of the first loss piece.

a) Estimation

Let $X_1, \ldots, X_n$ be the simulated independent draws from the portfolio loss distribution. The portfolio shortfall S_α at the confidence level α can be consistently estimated by the respective conditional mean of the empirical loss distribution. Let

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$$

be the order statistic of the random draws from the loss distribution, and let

$$c(\alpha, n) := \begin{cases} \alpha n & \text{if } pn \in \mathbf{N} \\ \lfloor \alpha n \rfloor + 1 & \text{else} \end{cases}.$$

Then the shortfall-estimator can be written as

$$\hat{S}_\alpha(X_1, \ldots, X_n) = \frac{1}{n - c(\alpha, n)} \sum_{i > c(\alpha, n)}^{n} X_{(i)}.$$

[277] Artzner et al., 1998, p. 20ff.

The consistency of the estimator follows from the theorem of Glivenko-Cantelli (Theorem 13 above).

b) Confidence intervals

It is difficult to directly calculate confidence intervals of the shortfall-estimator given above. We, therefore, simulate the distribution of the estimator.

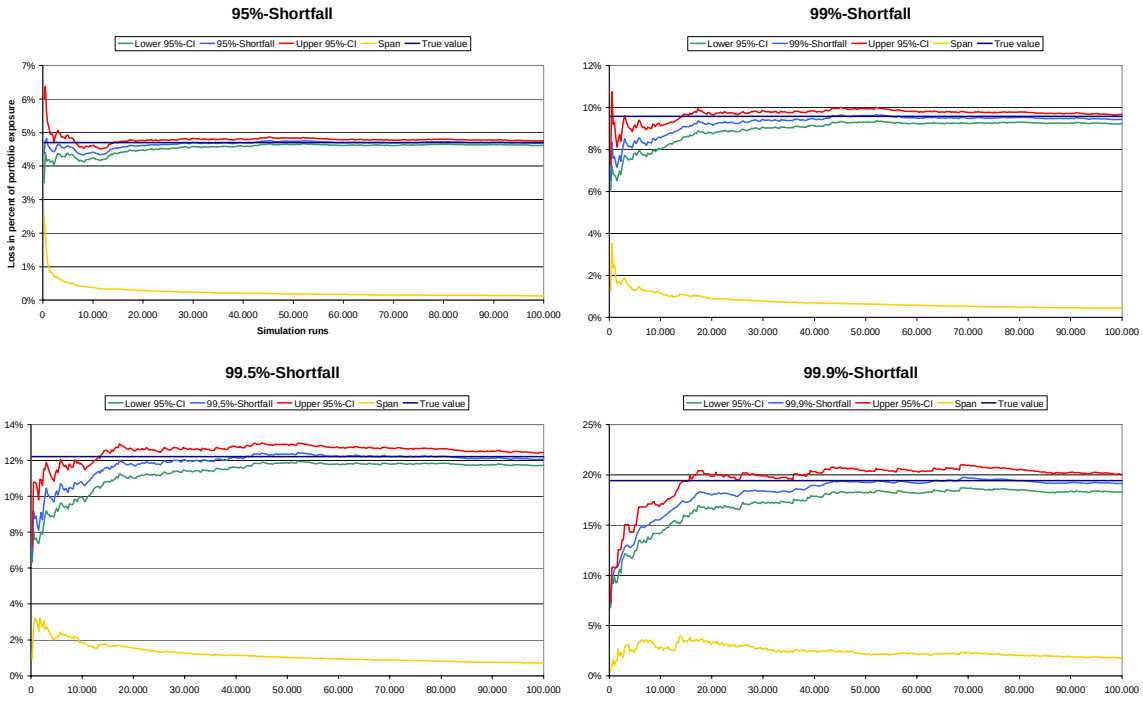


Figure 67: Accuracy of the estimation of the shortfall of portfolio losses dependent on the number of simulation runs

Figure 67 displays the simulated results for the portfolio shortfall at various confidence levels and confidence bands for the estimator at the 95%-confidence level for a homogenous portfolio with a default probability of 0.5% and risk index correlations of 30% in the normal correlation model. The distribution of the estimator was simulated 1,000 times and the mean, the 2.5%-percentile and the 97.5%-percentile were plotted [278].

Contrary to the expected loss and similar to the value at risk the shortfall-estimator is consistent, but not unbiased for high percentiles and low numbers of simulation runs. Already the 99%-shortfall requires at least 50,000 simulation runs in order to get an approximately unbiased estimation. This shows that measures of extreme tail risk such as the shortfall may need considerably more runs than measures of moderate tail risk such as the value at risk. This might especially cause problems when large portfolios are analyzed when the required num-

[278] Two simulations were performed sequentially. First, a sample of *n* points was drawn from the portfolio loss distribution and the shortfall was estimated as described above. Then, this was repeated 1,000 times in order to obtain 1,000 independent estimates for the shortfall as an empirical proxy for the distribution of the estimator.

ber of simulation runs cannot be performed within an acceptable amount of time. It is left for future research to develop more efficient estimators of shortfall risk that overcome these difficulties.

D. Risk analysis and risk management

Having quantified portfolio risk under several respects on the aggregate level for the portfolio as a whole, for a proactive risk management, i.e. for a risk management that methodically seeks opportunities and pitfalls and acts early, it is essential

- to understand and visualize portfolio structures,
- to compare portfolio components as to their contribution of risk and positive occasion to the portfolio,
- and to receive some hints by what actions portfolio risk can be reduced and how its composition from the risk management's point of view and its profitability can be improved.

In the subsequent sections, we will describe the concepts of marginal risks and expected risk adjusted returns. We will also show how they can be combined with the portfolio's exposure distribution to construct further risk measures, to define various types of limits, and to display the results in a 'risk management cockpit'. A risk management cockpit is a set of graphical visualizations that summarize the results and outline the portfolio components that need the risk manager's immediate attention. Finally, we state an algorithm how the portfolio shortfall can be minimized conditional to the portfolio's expected risk adjusted return.

To illustrate the notion of risk contributions and the analysis and risk management techniques, we analyze successively an example portfolio in the whole section. The example portfolio is almost homogenous[279] in the normal correlation model. It comprises clients in ten rating grades who have the following default probabilities and exposures:

	Rating I	II	III	IV	V	VI	VII	VIII	IX	X
Default Probability	0,03%	0,05%	0,09%	0,30%	0,50%	1,20%	3,10%	6,00%	7,50%	10,00%
Unsecured Exposure	24	5	12	17	28	18	11	19	7	5

Table 11: Definition of the example: default probabilities and exposures

[279] See above section II.A.2.a)(4).

Losses given default are set to 100%. Risk index correlations are assumed to equal 20%. For simplicity, but without real loss in generality, it is assumed that the only type of financial products consists in one year zero bonds[280]. The rating grades are defined as the portfolio segments of interest.

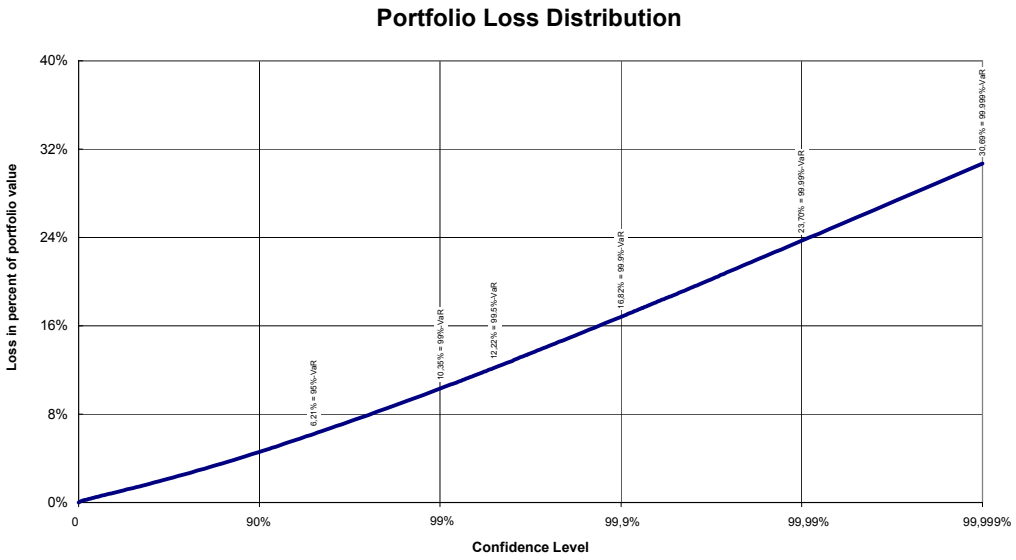


Figure 68: Loss distribution of the example portfolio

1. Marginal risks

For any risk measure, a portfolio segment's marginal risk is defined as its contribution to the total value of the risk measure for the entire portfolio. Thus, for instance, a segment's marginal 99%-value at risk is calculated as follows:

- Calculate the 99%-value at risk V_1 for the entire portfolio.
- Remove the segment from the portfolio.
- Recalculate the 99%-value at risk V_2 for the remaining portfolio.
- The segment's marginal 99%-value at risk is then the difference V_1 - V_2 of both values.

a) Marginal risks conditional to default

The simplest concept of risk contributions are marginal risks conditional to default of a specified portfolio segment. In this case, the segment's risk contribution is identical with its (unsecured[281]) exposure, i.e. its absolute loss given default.

280 The types of financial products that are traded by a bank affects only the methodology how the fair risk premium is to be calculated. It does not affect the calculation of expected risk adjusted returns and the risk management techniques.

281 In many models it is assumed that an exposure's secured part cannot be lost, even in case of default of the creditor. In practice, however, it frequently occurs that the value of securities turns out to be much lower than expected. In conse-

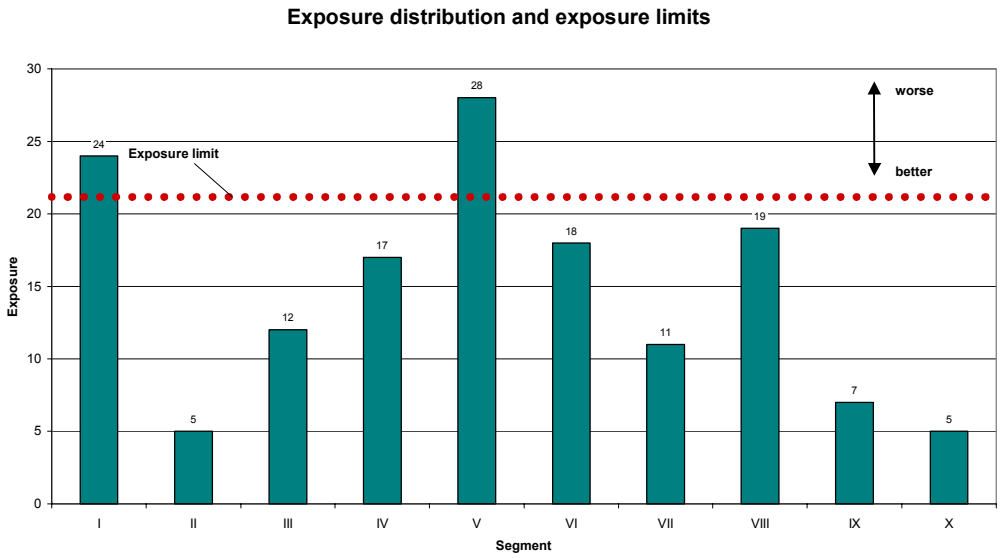


Figure 69: Exposure distribution and exposure limits

Figure 69 shows the exposure distribution of the example portfolio and indicates a possible exposure limit. Note that since a segment's exposure is its marginal risk conditional to default, the results are independent of clients' default probabilities. Exposure limits are, therefore, equivalent to the statement

"I don't want to lose more than X € if one segment defaults."

In the example, segment 1 and segment 5 exceed the set limit. As a consequence, the risk manager would have to inform the bank's management that exposures in both segments should be reduced.

b) Marginal risks prior to default

Marginal risks prior to default or simply marginal risks have many more facets than mere exposures. It is important to note that a segment's marginal risk does not only depend upon creditworthiness and exposure of the segment itself, but very importantly also on the other components of the portfolio if the portfolio standard deviation, the value at risk or the shortfall are used as basic risk measures in the analysis. The concept of marginal risk is, therefore, suited to analyze the risk structure of a portfolio and to localize areas of high risk concentrations and others that are better diversified in the portfolio.

quence, many banks account for the entire exposure as a client's or a segment's marginal risk conditional to default. For the same reason, the loss given default on the unsecured exposure usually is set to 100% in this analysis.

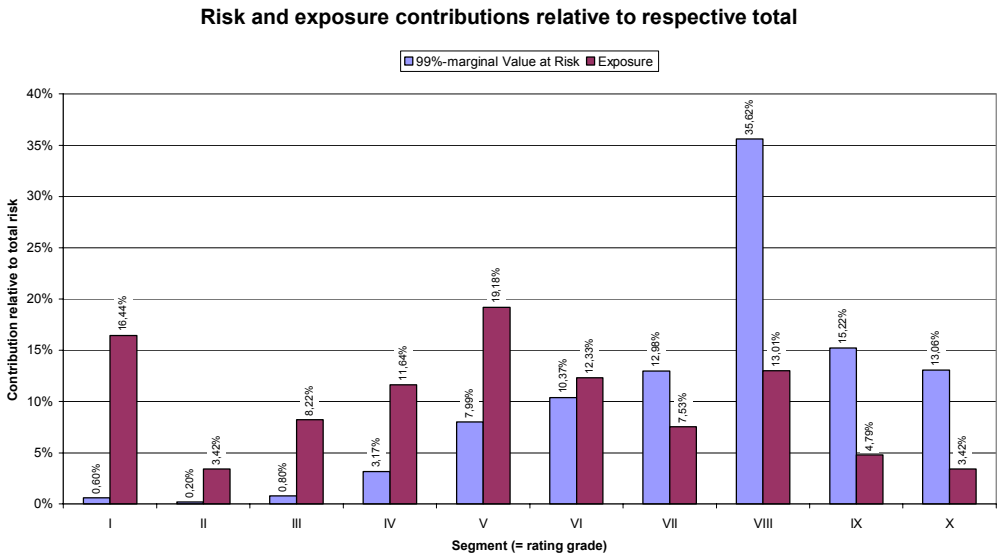


Figure 70: Risk and exposure concentrations

Figure 70 compares segments' concentrations[282] of exposures and of the 99%-marginal values at risk. It is evident that risk and exposure concentration greatly differ for almost all segments. Risk concentrations are much lower than exposure concentrations particularly for segments with high creditworthiness[283]. For instance, segment 1 contributed 16.44% to portfolio exposure and even exceeded an exposure limit, but only 0.6% to portfolio risk due to its extremely low probability of default. The opposite is true for segments with low creditworthiness. Segment 8 represents 13.01% of portfolio exposure, but not less than 35.62% of total portfolio risk. In other words, approximately 1/8th of portfolio exposure may stand for 1/3rd of portfolio risk.

[282] In this analysis it is assumed that the set of all segments is a partition of the total portfolio. A segment's risk concentrations in the portfolio is then defined as its marginal risk divided by the sum of all marginal risks. Exposure concentrations are defined respectively. Note that the concept of risk concentrations can also be used to distribute economic capital to portfolio components. This is particularly so because they sum up to 1.

[283] Segments are identical with rating grades in the example.

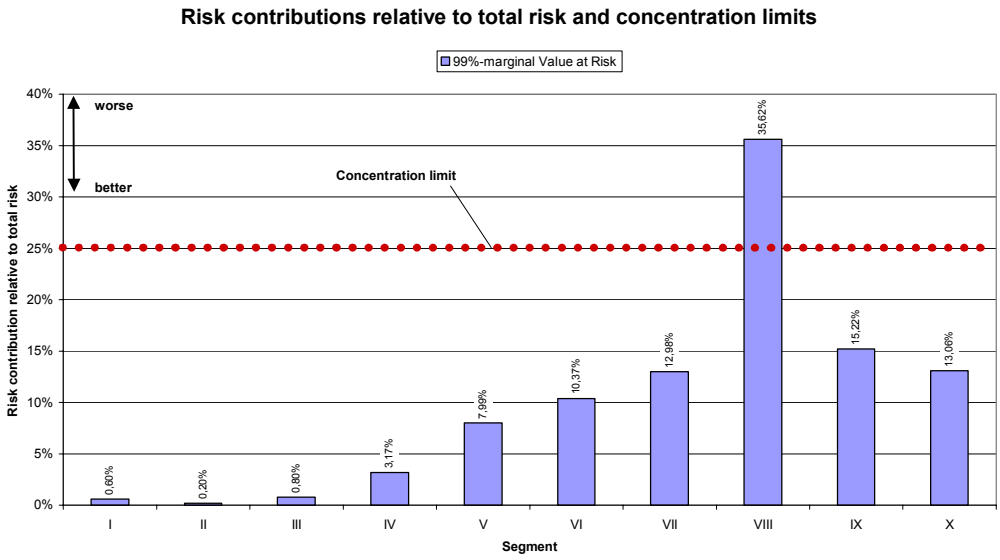


Figure 71: Risk concentrations and concentration limits

Similar to losses given default also risk concentrations can be limited. Here, the limit corresponds to the conviction

"I don't want to concentrate more than X% of the total portfolio risk in one segment."

Note that segment 1 and 5, that surpassed the exposure limit, are both far away from going beyond the concentration limit. On the other hand, exposure should be reduced in segment 8 in favor of other segments.

Complementary to risk concentrations, absolute marginal risks can be compared and limited (Figure 72).

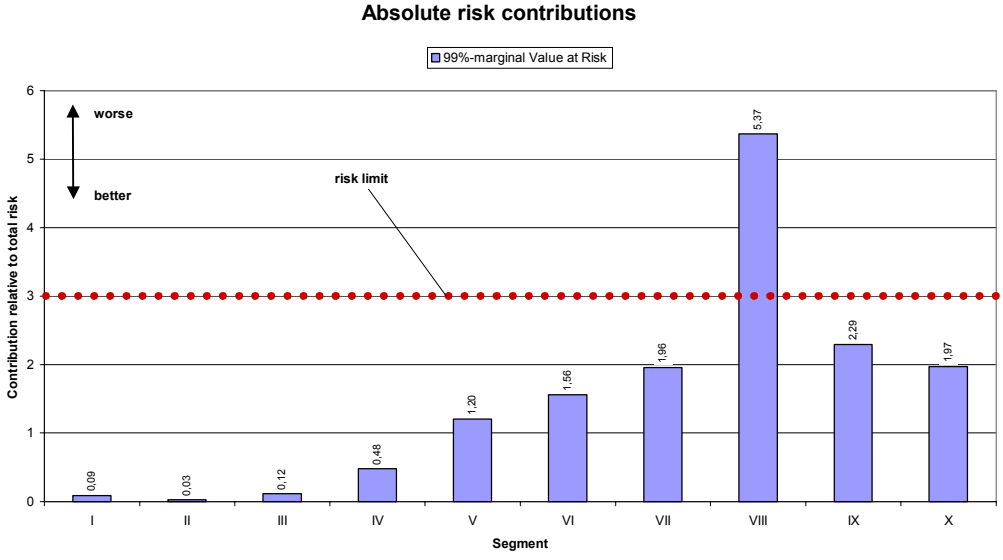


Figure 72: Absolute risk contributions and risk limits

Absolute risk contributions may, however, increase with the size of the portfolio. This is particularly true for the portfolio standard deviation as basic risk measure, which monotonously increases in portfolio exposure if default correlations are positive[284]. Less systematically, similar effects can occur if other risk measures are used. Limits to risk contributions are, therefore, much more difficult to interpret and have to be adjusted from time to time if the size of the portfolio changes.

c) Combinations of exposure and marginal risk

So far only simple risk measures were considered. However, sometimes it can be helpful to combine risk measures to get a more complete picture of the situation.

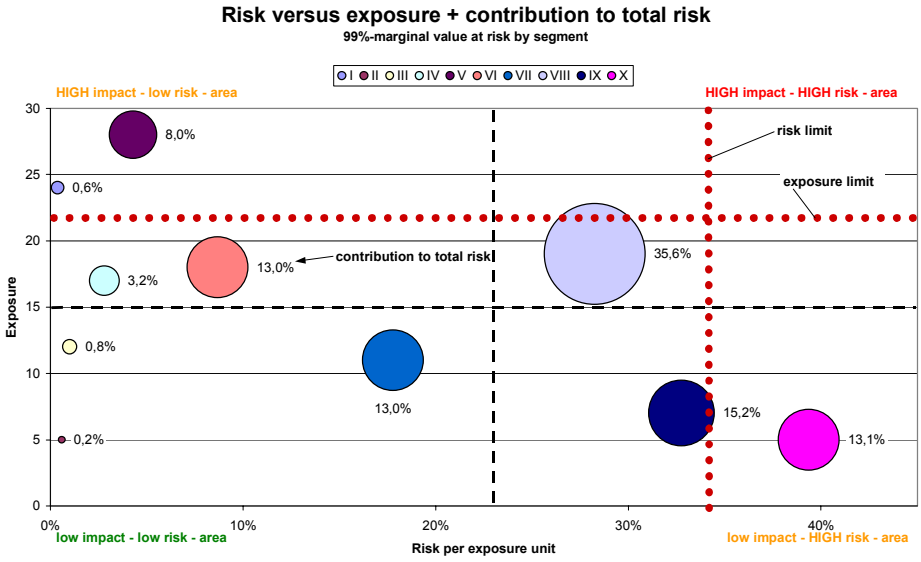


Figure 73: Risk versus exposure and risk concentration

Figure 73 displays three quantities at a time: on the axies the segments' absolute exposure versus their marginal risk contribution per exposure unit. The size of the bubbles indicates the risk concentration in each segment.

The segments of interest for the risk manager can be identified in several ways. Firstly, by the size of the bubbles. This analysis boils down to the same result as Figure 71.

Secondly, segments can be compared by absolute exposure and exposure limits can be applied. We see even more clearly than in Figure 69 that segment 1 and 5 are well above the set limit.

[284] Positive default correlations are the standard case in all of the quoted portfolio models.

Thirdly, segments can be contrasted by risk per exposure unit. Particularly, risky segments can be ruled out by the definition of risk limits. In our example, segment 10 clearly is in excess of the outlined limit.

Fourthly, all three features can be evaluated simultaneously. For this the chart can be divided in four quadrants. The lower left quadrant contains the small exposures that are subject to relatively low risks. Segments in this sector are inconspicuous. Moreover, limits and credit production could even be increased in the particular segments[285].

The upper left and the lower right quadrants correspond to the yellow traffic light. They comprise of respectively large exposures of relatively low risks and small, but risky exposures. Segments in these areas do not need urgent action, but they need close attention. This is more immediate if a segment represents a large risk-concentration in the portfolio. Segments 6 and 9 certainly need to be watched more carefully than segments 1 or 4. In this context, the reporting of risk-concentrations does not only supply an information that is valuable in itself, but it helps to prioritize the risk manager's monitoring activities and the establishment of both a watch list and an early warning system. In practice, the detection of imminent problems of segments in this area can lead to some action such as the reduction of individual limits, the hint to the bank's sales unit not to produce more exposure in the respective segments or the sale of parts of a segment's risk with the help of a credit derivative.

The upper right quadrant contains the explosives in the portfolio. It comprises the large and risky exposures where serious losses for the bank can be expected at any moment. Segments in this area require immediate action such as the instant reduction of exposure limits preventing the production of new credits in the respective areas, the selling of risk via credit derivatives and the obligation of the affected clients to supply additional securities or guaranties.

d) Expected risk adjusted returns

Limit management in the sense of limit supervision is reactive by definition. A methodical risk analysis and a functioning early warning system as discussed in the previous section are already much more proactive because they frequently indicate which limits should be redefined and supply background information that has a certain relevance for the management of a bank's credit production.

In order to establish a full-fledged proactive risk management that systematically seeks opportunities and tries to increase a bank's profitability at the same time as to improve the portfolio

[285] In practice, the size of the limits may vary with the segment, of course, although not symbolized in our example.

structure it is necessary to also include the segments' expected risk adjusted return into the examination.

A contract's expected risk adjusted return is defined as the contract's market interest rate less its fair risk premium[286] for a target return on equity that is equal to the long term risk free rate. If the return on equity equals the long term risk free rate, the credit's expected risk adjusted return on economic capital[287] can be zero and the long term risk free rate is still earned on the economic capital, i.e. the credit has a net profitability of zero. A segment's risk adjusted return then simply is the average of the risk adjusted returns of all contracts belonging to the segment weighted with each contract's contribution to the segment's total exposure.

Refinance Rate	Long Term Risk Free Rate	Target Return on Equity
4,75%	8,50%	25%

Rating	I	II	III	IV	V	VI	VII	VIII	IX	X
Market Rates	5,00%	5,10%	5,25%	5,90%	6,35%	7,80%	11,25%	12,00%	12,40%	12,60%

Table 12: Definition of the example: relevant interest rates

Table 12 states the values of the relevant interest rates for the analysis in our example. Market rates refer to a one-year zero bond with a counterparty having rating 1-10. Our bank can lend one-year-money at the refinance rate. Equity is invested at the long term risk free rate. Granted loans are intended to earn the difference to the target return on equity[288].

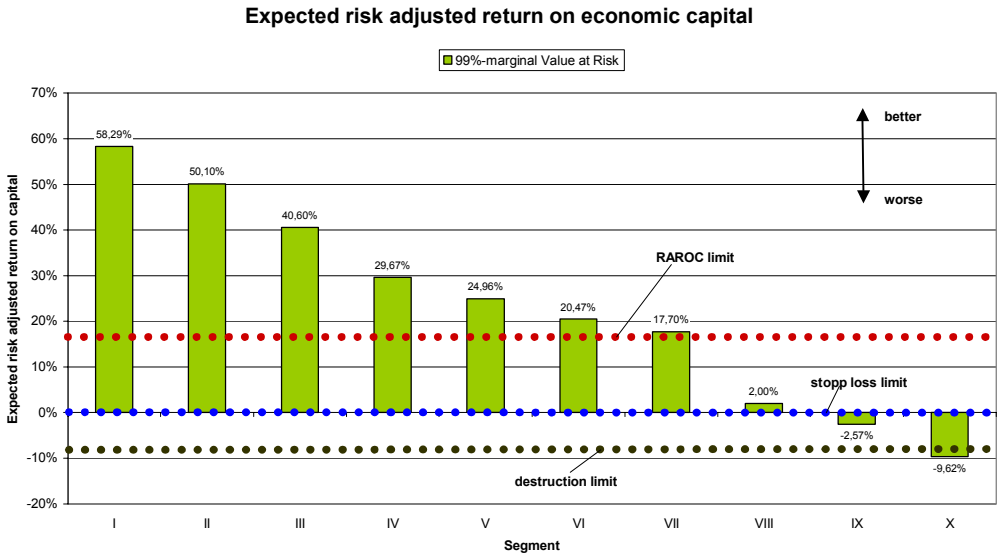


Figure 74: Expected risk adjusted return on economic capital

286 For the calculation of the fair risk premium see above section I.E.1.

287 The economic capital is defined as the capital that is supplied by the bank to cover the contract's risk. We suppose that the 99%-marginal value at risk is used to calculate economic capital instead of regulatory approaches.

288 See section I.E.1 above for more details on the role of the interest rates and all calculations. Particularly, we assume that on the one hand the whole credit volume is refinanced by leverage capital and that on the other hand the entire equity is invested at the long term risk free rate so that lent out money only has to earn the difference between target return on equity and the long term risk free rate.

Figure 74 shows the results for the expected risk adjusted return on economic capital (RAROC) for the example portfolio in the various segments and three types of limits. The RAROC limit marks the level of RAROC where the bank meets the target return on equity exactly. The stop loss limit stands for a RAROC of zero. Finally, the destruction limit is equal to the negative long term risk free rate.

As a consequence, four different areas can be distinguished in this chart. Firstly, a portfolio segment with a RAROC above the RAROC limit earns a higher return on equity than the intended target return. These segments are most profitable and, therefore, of particular interest for a proactive risk management. In our example, segment 1 to 7 belong to this group.

Secondly, segments with a RAROC between the RAROC limit and the stop loss limit still increase the bank's return on equity above the long term risk free rate, although their profitability is too low to meet the target return on equity. It is still better to invest in segments in this area than to stop granting credits at all if counterparties in the first group are not available as creditors. Segment 8 falls in this area in the example.

Thirdly, below the stop loss limit, but above the destruction limit the RAROC is negative implying that the bank's return on equity is reduced below the long term risk free rate, but still remains positive. Since a safe profit is lost it is not worthwhile investing in segments in this area. Segment 9 embodies this group in the example.

Fourthly, below the destruction limit, a segment's RAROC is so low that it fully absorbs the payoff of the investment of equity at the long term risk free rate. In this area, the bank expects to lose money from business with the respective segments, segment 10 in the example, and should immediately stop producing new contracts.

In the example, additional business with segment 9 and especially segment 10 is not desirable. Instead, credit production should be directed to segment 1 to 7 and, perhaps, segment 8 if necessary.

Once again, the expected risk adjusted return on capital is not the whole story since the size of accepted risk necessary to produce a certain RAROC may vary significantly between portfolio segments. It is, therefore, valuable to plot the expected risk adjusted return per exposure unit against the accepted risk per exposure unit (see Figure 75).

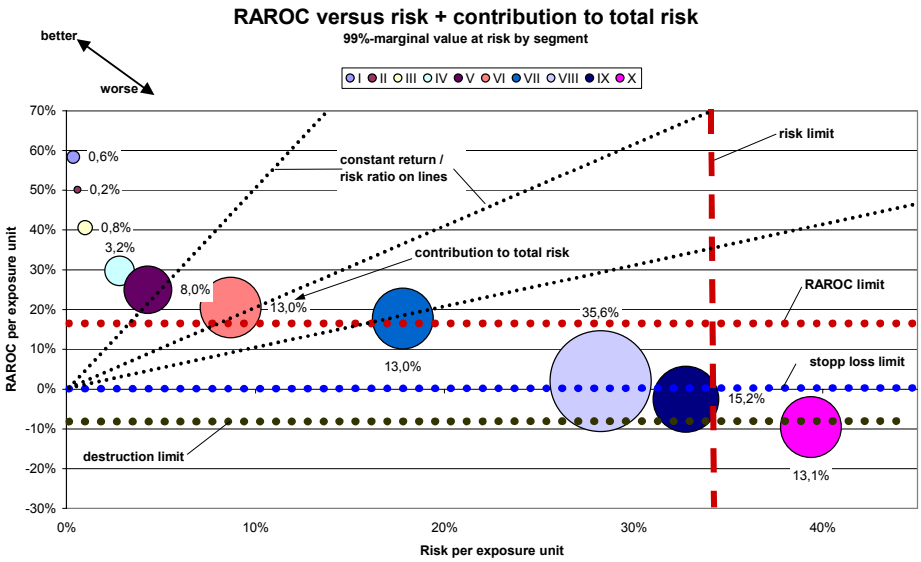


Figure 75: Accepted risk versus expected risk adjusted return per exposure unit and contribution to total risk

In the exhibit, RAROC, risk and risk concentration are combined showing that segment 10 misses several limits at a time. Conversely, segments 1 to 5 appear particularly favorable from the point of view of satisfying all three criteria simultaneously. Segments 6 and 7 are still above the RAROC limit, but carry a medium sized risk per exposure unit and already represent considerable risk concentrations in the portfolio.

e) Summary

It has become clear from the discussion above that credit portfolio analysis is a multidimensional task. Being unobtrusive or even positive from one point of view, a segment may be less or even undesirable from another.

In the example, segment 1, having the highest credit worthiness, has the most favorable RAROC, carries very little risk per exposure unit and represents only a small risk concentration in the portfolio so that one would want to enlarge business in that area. Unfortunately, segment 1 already exceeds the exposure limit, though, so that additional credit production is impossible here. Although less pronounced, the same argument holds true for segment 5.

On the other hand, segments 8 to 10 are so risky and so little profitable that credit granting should be avoided in this area. This is especially so as a result of the poor RAROC of segments 9 and 10, the additionally enormous risk concentration in segment 8 and the surpassing of the risk limit by segment 10.

This analysis leads to the following actions:

- Stop credit production in segments 1, 5, 8, 9, 10.

- Reduce risk in segments 8 and 10.
- Increase credit production in segments 2, 3, 4, 6, and 7, particularly in segments 2, 3, and 4, less so in segments 6 and 7.

2. Credit derivatives

It is a typical problem for a credit portfolio manager that there are portfolio segments where risk is undesirably strongly concentrated or surpasses risk limits so that he would like to reduce risk in the respective areas to rebalance the portfolio. In this situation, credit derivatives can be applied. 'Credit derivative' is an umbrella term for all kinds of insurance contracts against losses from credit risks. They enable a bank to transfer the credit risk from a specific contract or a portfolio of contracts to another bank or a group of banks.

We don't want to discuss the design and application of credit derivatives in detail. We rather want to state two essential prerequisites for the successful appliance of credit derivatives in the interbank risk trade.

Firstly, in order to sell the risk of the right portfolio components, a risk manager has to have the necessary implements to quantitatively analyze the portfolio he is responsible for.

Secondly and more importantly, the quantitative and qualitative structure of the sub-portfolio whose risk is to be transferred has to be transparent for all parties that participate in the credit derivative so that the contract can be objectively valued. This is particularly true for the assessment of default probabilities of counterparties in the sub-portfolio. The risk selling bank has to make its internal rating results accessible for the risk taking bank and also has to prove that its rating methodology is conceptually sound so that an objective benchmark price for the derivative can be calculated. This transparence requirement is a central side condition when a new internal rating scheme is implemented.

3. Portfolio optimization

A complementary approach to improve portfolio quality is algorithmic portfolio optimization under certain constraints such as non-negativity of exposures, the size of expected returns or others. This was long an unsolved problem because the standard approach of minimizing portfolio variance under side-constraints was difficult to apply to credit portfolios due to the lack of interpretability of portfolio variance (or standard deviation). However, in their 1999 semi-

nal paper Stanislav Uryasev and Tyrell Rockafellar[289] proposed a methodology to minimize portfolio shortfall[290] under the respective side-constraints.

a) Optimization approach

Indicate clients (or portfolio segments of interest[291]) by i for $i = 1,..., n$. Let $X = (X_1,..., X_n) \in \Omega \subset \mathbf{R}^n_{\geq 0}$ be the vector of clients' exposures in the portfolio chosen out of a certain subset $\Omega \subset \mathbf{R}^n_{\geq 0}$. In the sense of the optimization problem, X is a decision vector reflecting the composition of the credit portfolio. Let $Y = (Y_1,..., Y_n) \in \mathbf{R}^n$ be a random vector that stands for the uncertainties of the future value of the exposures. For ease of exposition we assume in the following that the distribution of Y is continuous with density $p(Y)$. In practice, the distribution of Y need not be known analytically, but is being simulated in accordance with the portfolio model used[292]. Let $f(X,Y) \in \mathbf{R}$ be the portfolio loss associated with the exposure vector X and the random 'creditworthiness' vector Y. For given X, the distribution of $f(X,Y)$ is the portfolio loss distribution.

Let further

$$\Psi(X,\alpha) := P\{f(X,Y) \leq \alpha\} = \int_{f(X,Y)\leq\alpha} p(Y)dY$$

be the probability that $f(X,Y)$ does not exceed a given threshold α.

For a given confidence level $\beta \in (0, 1)$ the β-VaR and the β-shortfall are

$$VaR_\beta(X) = \min\{\alpha \in \mathbf{R} : \Psi(X,\alpha) \geq \beta\}$$

and

$$S_\beta(X) = \frac{1}{1-\beta} \int_{f(X,Y)\geq VaR_\beta(X)} f(X,Y)p(Y)dY \,.$$

289 Rockafellar and Uryasev 1999.

290 Minimization of portfolio shortfall is not equivalent to minimization of portfolio value at risk. The shortfall always dominates the respective value at risk.

291 In practice, it may be difficult to do a portfolio optimization at the client level because single clients' exposures usually cannot be modified easily without the respective client's agreement. For this reason, it is rather more practicable to consider portfolio segments at an intermediate level of aggregation in the portfolio where exposures can be more freely adapted.

292 Note in particular, that the methodology suggested by Rockafellar and Uryasev is independent of the actual portfolio model that is used by a financial institution to assess portfolio risk. The only place where the portfolio model enters in is the simulation of clients' future creditworthiness.

The key to the optimization approach of Rockafellar and Uryasev is the characterization of optimal shortfall and the corresponding value at risk in terms of the function F_β defined as

$$F_\beta(X,\alpha) = \alpha + \frac{1}{1-\beta} \int_{Y \in \mathbf{R}^n} [f(X,Y) - \alpha]^+ \, p(Y) dY \, .$$

The relationship between optimal shortfall, corresponding value at risk and the function F_β is captured in the following

Theorem 15 (Rockafellar and Uryasev 1999):
Minimizing the β-shortfall of the loss associated with X over all $X \in \Omega$ is equivalent to minimizing $F_\beta(X,\alpha)$ over all $(X,\alpha) \in \Omega \times \mathbf{R}$, in the sense that

$$\min_{X \in \Omega} S_\beta(X) = \min_{(X,\alpha) \in \Omega \times \mathbf{R}} F_\beta(X,\alpha).$$

A pair (X^*,α^*) achieves the second minimum if and only if X^* achieves the first minimum and $\alpha^* = VaR_\beta(X^*)$.

According to the theorem, it is sufficient to do the minimization of the β-shortfall with the function F_β that does not contain the β-VaR, which often is mathematically troublesome to handle, instead of using the β-shortfall directly. The theorem also states that the value at risk corresponding to the optimal β-shortfall is automatically given as one of the calculation results, quasi as a byproduct of the optimization.

In practice, the distribution of Y in F_β can be approximated by a sampled empirical distribution due to the theorem of Glivenko-Cantelli. With simulation results $Y_1,...,Y_m$ the approximation $\widetilde{F}_\beta$ of F_β can be defined as

$$\widetilde{F}_\beta(X,\alpha) = \alpha + \frac{1}{(1-\beta)m} \sum_{j=1}^{m} [f(X,Y_j) - \alpha]^+ \, .$$

b) A portfolio optimization

As an example, we consider a heterogeneous portfolio of 10,000 clients[293] in n = 20 segments with a total portfolio exposure of € 1,000 in the normal correlation model[294]. Risk index correlations were set to 30%. The portfolio loss distribution was sampled m = 2,000 times[295].

[293] Individual default probabilities, exposures and segment adherence were assigned randomly.

relations were set to 30%. The portfolio loss distribution was sampled m = 2,000 times[295]. Exposures and losses were aggregated per segment in each simulation run so that the optimization can be performed at the segment level[296]. The coefficients of the random vector $Y_j = \left(Y_j^1,...,Y_j^n\right) \in \mathbf{R}^n$ signify which fraction of the portfolio exposure in the respective segment was lost in simulation run j for $j = 1, \ldots, m$. The portfolio loss function f then simply is

$$f(X,Y) = X^t Y$$

so that the optimization problem can be stated as

$$\min_{X\in\Omega,\alpha\in\mathbf{R}} \widetilde{F}_\beta(X,\alpha) = \min_{X\in\Omega,\alpha\in\mathbf{R}} \alpha + \frac{1}{(1-\beta)m}\sum_{j=1}^{m}\left[X^t Y_j - \alpha\right]^+ .$$

With $U = (U_1,...,U_m)$ this is the same as

$$\min_{X\in\mathbf{R}^n,\alpha\in\mathbf{R},U\in\mathbf{R}^m} \alpha + \frac{1}{(1-\beta)m}\sum_{j=1}^{m} U_j$$

subject to

$$\begin{aligned} &X \in \Omega, \\ &U_j \geq X^t Y - \alpha, \\ &U_j \geq 0 \end{aligned}$$

for $j = 1, \ldots, m,$ so that the task is reduced to a linear optimization problem[297].

The feasibility set $\Omega \subset \mathbf{R}^n_{\geq 0}$ was specified as a linear return-constraint[298]

$$r^t X \geq \theta \cdot \sum_{i=1}^{n} X_i$$

where $r = (r_1,...,r_n) \in \mathbf{R}^n$ is the vector of expected returns per exposure unit in each segment for fixed values of $\theta > 0$ and the trivial constraint

$$X_i \geq 0$$

294 The normal correlation model was used as a pure default model so that rating migrations other than to default were ignored.

295 The optimization was performed with an own implementation of the simplex algorithm. It turned out that the algorithm could not handle problems with more than about 2,000 constraints, thus, limiting the number of simulation runs (see below). 2,000 runs are, however, too few to precisely forecast the 95%-shortfall. Optimization results are, therefore, rather illustrative in nature. A serious implementation would need a high-end linear solver such as IBM-OSL or CPLEX.

296 We suppose that the composition of the segments remains unchanged by the optimization.

297 Note that the simplex algorithm assumes non-negativity of all variables so that the number of non-trivial constraints is approximately equal to the number of simulation runs.

298 Other possible linear constraints are exposure limits or exposure concentration limits. Risk or risk concentration limits usually are non-linear in nature.

for $i = 1, \ldots, n$.

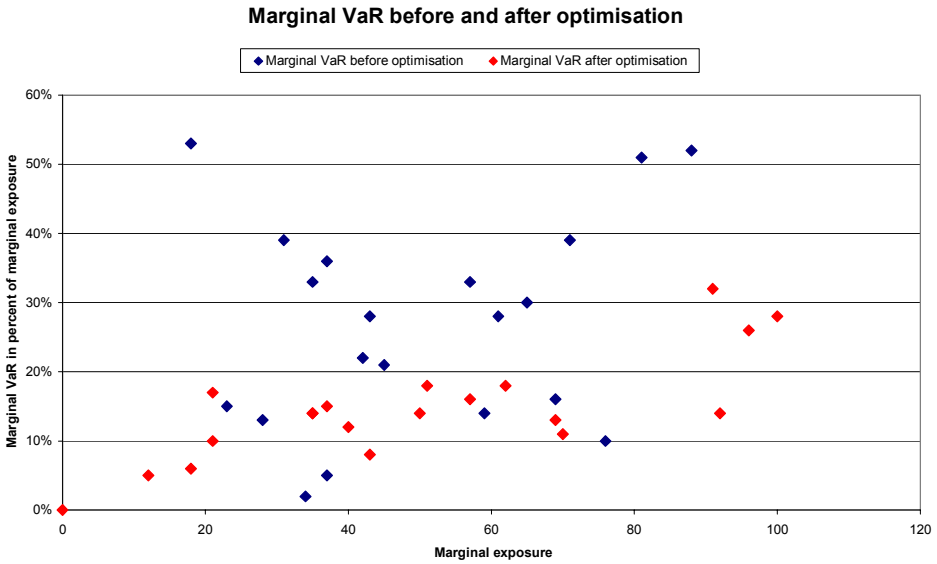


Figure 76: Marginal VaR before and after optimization

Figure 76 shows the segments' marginal values at risk before and after optimization of the 95%-portfolio shortfall. It is particularly striking that after the optimization the number of small exposures with high marginal risks has decreased substantially. Note moreover that one segment has an exposure of zero after the optimization and has completely dropped out of the portfolio.

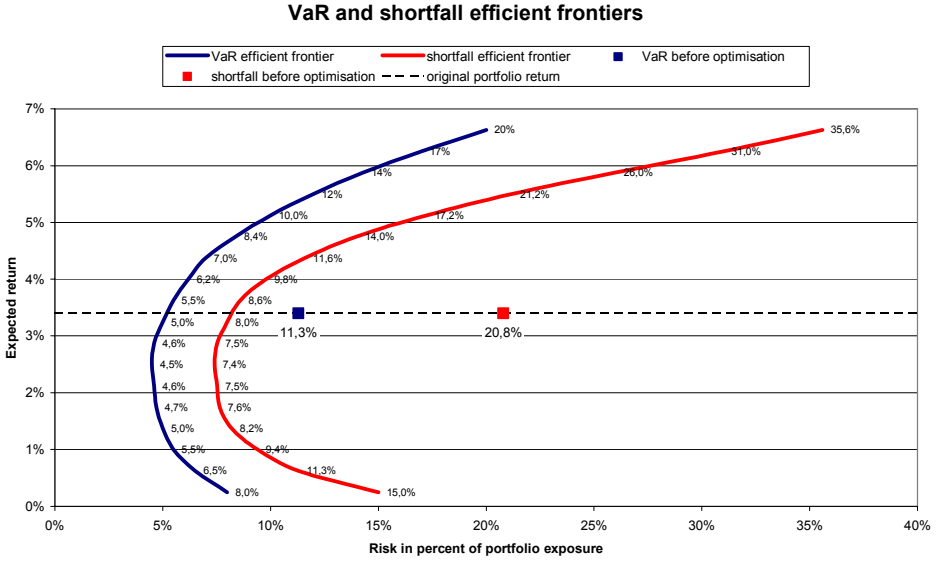


Figure 77: VaR and shortfall efficient frontiers

Figure 77 illustrates the efficient frontiers[299] of the example portfolio for different values of the minimum expected return θ. It turns out that the optimization had a remarkable effect on the portfolio shortfall and also on the portfolio value at risk by reducing risk by more than half in both cases. Overall, the optimization greatly improved portfolio quality both from the point of view of portfolio structure and total portfolio risk.

Conclusion

The discussion has shown that credit risk modeling and management is a complex task where many different aspects play an important role and where many errors can be committed. The analysis has identified some of these errors, particularly in the estimation of default probabilities and the concepts of dependence among clients. Most importantly, the thesis proposes solutions for some of the deficiencies of current risk models.

Moreover, the entire risk management process has been assessed and methods for portfolio analysis and management have been described in detail. We are convinced that some of our results can help financial institutions to avoid some of the pitfalls of credit risk modeling and management and to improve the quality of their risk management methods and in turn the credit risk quality of their portfolios.

[299] Only the portfolio shortfall was optimized. The values for the value at risk just correspond to the respective shortfalls and may not be optimal for a value at risk criterion. Experiments with various portfolios also showed that the value at risk efficient frontier is not necessarily convex. Moreover, we also found examples where the portfolio value at risk efficient frontier did not even fall into a part that was monotonously decreasing and another that was increasing in the expected return. In practice, the VaR-efficient frontier can take a great variety of shapes.

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